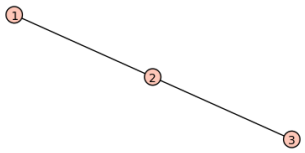
Last name _____

First name _____

LARSON—MATH 756—HOMEWORK WORKSHEET 12
Knuth's *Sandwich Theorem* paper



We will show that $\vartheta_1(P_3) \leq \vartheta_2(P_3)$ for the path graph P_3.

1. We'll guess that $\vartheta(P_3) = 2$. Find a *feasible* matrix A for P_3 with largest eigenvalue 2. Explain why this shows $\vartheta_2(P_3) \leq 2$.
2. Find all the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of your A and corresponding normalized eigenvectors for P_3.
3. Use these to write $A = QDQ^T$, for an orthogonal matrix Q and diagonal matrix D.
4. Let λ be the largest eigenvalue of A. Find $B = \lambda I - A$.
5. Check that the eigenvectors of A are also eigenvectors of B and that the eigenvalues of B are $\lambda - \lambda_1$, $\lambda - \lambda_2$, $\lambda - \lambda_3$.
6. Let $X = diag(\sqrt{\lambda - \lambda_1}, \sqrt{\lambda - \lambda_2}, \sqrt{\lambda - \lambda_3})Q^T$. Check that $B = X^T X$.
7. Let the columns of X be $\hat{x}_1, \hat{x}_2, \hat{x}_3$. Check that $B_{i,j} = \hat{x}_i \cdot \hat{x}_j$ for each i, j.
8. Let $\hat{a}_i = \frac{1}{\lambda} \begin{bmatrix} 1 \\ \hat{x}_i \end{bmatrix}$. Find $\hat{a}_1, \hat{a}_2, \hat{a}_3$.
9. Check that the $\hat{a}$ vectors are an orthonormal representation for P_3.
10. Check that $\max_{v \in V(P_3)} \frac{1}{c(\hat{a}_v)} = 2$.
11. Explain why $\vartheta({}_1P_3) \leq 2$.
12. Explain why $\vartheta_1(P_3) \leq \vartheta_2(P_3)$.

13. Write up a nice, clean proof that $\vartheta_1(G) \leq \vartheta_2(G)$ for any graph G. Be verbose.