Last name _____

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LARSON—MATH 756—CLASSROOM WORKSHEET 23
Knuth—Sandwich Theorem

Review

1. What is the main idea of the proof that $\vartheta_1 \leq \vartheta_2$?
2. If A is a symmetric matrix with largest eigenvalue λ, what are the eigenvalues of $B = \lambda I - A$? (Why is B a *semidefinite* matrix?)
3. Show: $\vartheta_1 \leq \vartheta_2$ for any graph.
4. Illustrate the ideas of the proof to the path graph P_3. Argue that $\vartheta_1(P_3) \leq \vartheta_2(P_3)$.

New; Proof that $\vartheta_2 \leq \vartheta_3$

For a graph G, ϑ_3 is defined to be

$$\vartheta_3(G) = \max\{\sum_{u,v} \hat{b}_u \cdot \hat{b}_v | \{\hat{b}_w\} \text{ is a normalized orthogonal labeling of } \bar{G}\}.$$

1. What is a *normalized* labeling?

2. Find an optimal normalized labeling of $\bar{P}_3$.

3. Let $B_{u,v} = \hat{b}_u \cdot \hat{b}_v$. Find B.

4. What can we say about symmetric positive semidefinite matrices?

5. Why must B be symmetric positive semidefinite, which is 0 when $u \nsim v$?

Fact. If C is a symmetric matrix such that $C \cdot A \geq 0$ for all symmetric positive semi-definite B with $B_{u,u}$ for $u \sim v$ then $A = X + Y$ where X is symmetric positive semidefinite and Y is symmetric and $Y_{v,v} = 0$ for all v and $Y_{u,v} = 0$ when $u \nsim v$.

6. Why is this **claim** true? If $\vartheta_3(G) = \vartheta_3$ and $\{b_w\}$ is any non-zero orthogonal labeling of $\bar{G}$ (not necessarily normalized) we have

$$\sum_{u,v} \hat{b}_u \cdot \hat{b}_v \leq \vartheta_3 \sum_v ||\hat{b}_v||^2.$$

7. Why does this say $J \cdot B \leq (\vartheta_3 I) \cdot B$ for all symmetric positive definite B with $B_{u,v} = 0$ when $u \sim v$ (and J is the all-1's matrix)?

8. Why does this imply that $\vartheta_3 I - J$ is a C matrix as in the **Fact**?

Then **Fact** says we can write $\vartheta_3 I - J = X + Y$, with X and Y as in the **Fact**.

9. Let $A = J + Y = \vartheta_3 I - X$. Why is A feasible for G?

10. Why is $\Lambda(A) \leq \vartheta_3$? And then why is $\vartheta_2 \leq \vartheta_3$?