Last name ______________________

First name ______________________

LARSON—OPER 731—CLASSROOM WORKSHEET 07
Linear Programs

Concepts

- (Sec. 2.1) *infeasible linear program*, *certificate of infeasibility*, *certificate of unboundedness*, *certificate of optimality*.
- (Sec. 2.2) SEF, *slack variables*.
- (Sec. 2.4) *basis*, *basic* variable, *nonbasic* variable, *basic solution*, *basic feasible solution*, *canonical form*.

Bases and Canonical Forms

1. If A is $m \times n$ and $j \in \{1, \ldots, n\}$, what is A_j? If $J \subseteq \{1, \ldots, n\}$, what is A_J?

2. If $B \subseteq \{1, \ldots, n\}$ when is B a *basis* for A? What can we say about A_B? What is N?

3. What are *basic* variables x_j? What are *nonbasic* variables?

4. What is a *basic solution* of $Ax = b$ for a basis B? (What is a *basic feasible solution*?)

5. What is the *canonical form* for an LP with respect to a basis B?

6. Solve using the simplex method.

1 Consider the LP problem $\max\{c^\top x : Ax = b, x \geq \mathbb{0}\}$, where

$$A = \begin{pmatrix} 1 & 2 & -2 & 0 \\ 0 & 1 & 3 & 1 \end{pmatrix} \qquad b = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \qquad c = \begin{pmatrix} 0 \\ 3 \\ 1 \\ 0 \end{pmatrix}.$$

(a) Beginning with the basis $B = \{1, 4\}$, solve the problem with the simplex method. At each step, choose the entering variable and leaving variable by Bland's rule.

Simplex Method Steps

$$max\{c^T x + \bar{z} : Ax = b, x \geq \mathbb{O}\}$$

(a) **Initialize** Find an initial basis and basic feasible solution.

i. Use Gaussian elimination to check if system is feasible and that rows are linearly independent.
ii. For $m \times n$ A, add add m extra columns for auxiliary system, with easy-to-find basic feasible solution.
iii. We'll have basis B, basic feasible $\bar{x}$ (with $\bar{x}_N = \mathbb{O}$), and value of the objective function for $\bar{x}$.

(b) **Put in Canonical Form**

i. A_B is necessarily invertible. Find A_B^{-1}.
ii. The new system is $A_B^{-1}Ax = A_B^{-1}b$ (and the "new A" is $A_B^{-1}A$, the "new b" is $A_B^{-1}b$, and the "new A_B" is I).
iii. Update the objective function so that the "new c" has $c_B = 0$). Add $0 = y^T b - y^T Ax$ to the old/existing objective function, and solve for y (where the "new c_B" is $\mathbb{O}$).

(c) **Improve, if possible**

i. Is any $c_k > 0$? If not, then **done**. The current objective achieves its maximum for $\bar{x}$.
ii. Let k be the smallest index with $c_k > 0$ (k will "enter the basis").
iii. Write $b = A_B x_B + x_k A_k$, and let $x_k = t$. Get $x_B = b - tA_k \geq \mathbb{O}$, and find the largest possible value for t.
iv. Let r be the corresponding row. Substitute this t into $x_B = b - tA_k$, to get updated $\bar{x}$ values (we must have $\bar{x}_r = 0$). Find the value of the objective function for this new $\bar{x}$.
v. Update the basis. The new basis is $B \setminus \{k\} \cup \{r\}$. For this "new B", we must have $\bar{x}_N = \mathbb{O}$.

(d) **Iterate** Go back to Step (b).

7. If we have a feasible LP, in SEF, how can we find an initial basis, and initial basic feasible solution?