Last name ______________________

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LARSON—OPER 731—CLASSROOM WORKSHEET 06
Linear Programs

Concepts

- (Sec. 2.1) *infeasible linear program*, *certificate of infeasibility*, *certificate of unboundedness*, *certificate of optimality*.
- (Sec. 2.2) SEF, *slack variables*.
- (Sec. 2.4) *basis*, *basic* variable, *nonbasic* variable, *basic solution*, *basic feasible solution*, *canonical form*.

Bases and Canonical Forms

1. If A is $m \times n$ and $j \in \{1, \ldots, n\}$, what is A_j? If $J \subseteq \{1, \ldots, n\}$, what is A_J?

2. If $B \subseteq \{1, \ldots, n\}$ when is B a *basis* for A? What can we say about A_B? What is N?

3. What are *basic* variables x_j? What are *nonbasic* variables?

4. What is a *basic solution* of $Ax = b$ for a basis B? (What is a *basic feasible solution*?)

5. What is the *canonical form* for an LP with respect to a basis B?

6. Consider the following LP in SEF:

$$\begin{aligned} \max \quad & (1, -2, 0, 1, 3)x \\ \text{subject to} \quad & \\ & \begin{pmatrix} 1 & -1 & 2 & -1 & 0 \\ 2 & 0 & 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ & x \geq \mathbb{0}. \end{aligned}$$

7. Check that $\{1, 4\}$ is a basis.

8. Find an equivalent LP in canonical form.

Simplex Method Steps

$$max\{c^T x + \bar{z} : Ax = b, x \geq \mathbb{O}\}$$

(a) Check that $Ax = b$ has linearly independent rows (using Gaussian elimination).

(b) Check that $Ax = b$ is feasible by solving the system (using Gaussian elimination) or discovering that the system is infeasible. Assume A is $m \times n$. You'll get m leading 1's. Let B be the indices of those 1's columns. The initial x can be taken to be the solution vector with 0's for all non-B components (so $x_N = \mathbb{O}$).

(c) We have initial basis B and initial basic feasible solution $\bar{x}$. Now write $Ax = b$ in canonical form with respect to B:

   i. On the first pass $A_B = I$ (due to Gaussian elimination). Otherwise left-multiply $Ax = b$ by A_B^{-1}.

   ii. Let $y = (y_1, ..., y_m)^T$. Then $y^T Ax = y^T b$, and $y^T b - y^T Ax = \mathbb{O}$. Add this to $c^T x + \bar{z}$ to get and updated objective function.

   iii. We want y so that our updated objective function has $c_B = 0$. The new objective function is $(c^T x + \bar{z}) + (y^T b - y^T Ax)$. Solve to get: $c_B^T x_B - y^T A_B x_B = \mathbb{O}$.

(d) Let c' be the x coefficients in the updated objective function (with $c'_B = \mathbb{O}$). Are any c' entries positive? If not we're done ($\hat{x}$ is optimal). Let k be the smallest index with $c'_k > 0$ (k is necessarily in N). We want to increase $\bar{x}_k$ (which is necessarily 0) to t.

(e) We'll keep all other $\bar{x}$ components for indices in N equal to 0. So, $b = A_B x_B + x_k A_k = x_B + tA_k$. So $x_B = b - tA_k \geq \mathbb{O}$, and $b \geq tA_k$.

(f) For each component b_i of b, we must have $\frac{b_i}{A_{ik}}$ (when $A_{ik} \neq 0$). Let $t = \min\{\frac{b_i}{A_{ik}} : i \in B\}$.

(g) Let r be the smallest index where $t = \frac{b_i}{A_{ik}}$.

(h) Let the updated feasible solution be $\bar{x}'$ with $\bar{x}' = b - tA_k$, $\bar{x}'_k = t$ (and $\bar{x}'_i = 0$ for $i \in N \setminus \{k\}$).

(i) Remove r from B and add k. So the new basis is $B' = B \setminus \{r\} \cup \{k\}$.

(j) Add r to N and remove k. So the new nonbasis is $N' = N \setminus \{k\} \cup \{r\}$.

(k) We now have new basis B', nonbasis N', basic feasible $\bar{x}'$ (with $\bar{x}'_N = \mathbb{O}$), and objective function with x coefficients c' (with $c'_B = \mathbb{O}$).

9. Solve.

1 Consider the LP problem $\max\{c^{\top}x : Ax = b, x \geq \mathbb{0}\}$, where

$$A = \begin{pmatrix} 1 & 2 & -2 & 0 \\ 0 & 1 & 3 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \quad c = \begin{pmatrix} 0 \\ 3 \\ 1 \\ 0 \end{pmatrix}.$$

(a) Beginning with the basis $B = \{1, 4\}$, solve the problem with the simplex method. At each step, choose the entering variable and leaving variable by Bland's rule.