Last name ______________________

First name ______________________

LARSON—OPER 731—CLASSROOM WORKSHEET 03
Linear Programs

Concepts

- (Sec. 1.3) *integer program*, *mixed integer program.*
- (Sec. 1.4) *st-path*, *matching*, *perfect matching.*
- (Sec. 2.1) *infeasible linear program*, *certificate of infeasibility*, *certificate of unboundedness*, *certificate of optimality*, SEF, *slack variables.*

Shortest Path Problem

1. Let $G = (V, E)$ be a graph with vertices s and t and edge lengths/costs/weights $(c_e : e \in E)$. How did we model finding a shortest path in from s to t in G with an integer program?

2. It remained to argue that an optimal solution to this IP corresponds to a shortest (minimum cost) st-path; why is that true?

3. What are the 3 possibilities for an LP?

Infeasible LPs

4. How can we show that the following system of constraints is infeasible? What happens if we left multiply by $y = (1, -2, 1)$?

$$Ax = b,$$

where

$$A = \begin{pmatrix} 4 & 10 & -6 & -2 \\ -2 & 2 & -4 & 1 \\ -7 & -2 & 0 & 4 \end{pmatrix} \quad b = \begin{pmatrix} 6 \\ 5 \\ 3 \end{pmatrix} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}. \tag{2.6}$$

5. **Claim**: If A is a matrix and b is a vector with $Ax = b$ $(x \geq 0)$, then the system has no solution if there is a vector y such that: (1) $y^T A \geq \mathbb{O}^T$, and $y^T b < 0$.

6. What is a *certificate of infeasibility*?

Unbounded LPs

$$\max\{z(x) = c^\top x : Ax = b,\ x \geq \mathbb{0}\},$$

where

$$A = \begin{pmatrix} 1 & 1 & -3 & 1 & 2 \\ 0 & 1 & -2 & 2 & -2 \\ -2 & -1 & 4 & 1 & 0 \end{pmatrix} \quad b = \begin{pmatrix} 7 \\ -2 \\ -3 \end{pmatrix} \quad c = \begin{pmatrix} -1 \\ 0 \\ 3 \\ 7 \\ -1 \end{pmatrix} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}. \tag{2.7}$$

7. Let $\bar{x} = (2,0,0,1,2)^T$, $d = (1,2,1,0,0)^T$, and $x(t) = \bar{x} + td$. Check that:
   (a) $\bar{x}$ is feasible.
   (b) $Ad = \mathbb{O}$.
   (c) $Ax(t) = b$.
   (d) $x(t) \geq \mathbb{O}$ for every $t \geq 0$.
   (e) $c^T x(t) \to \infty$ as $t \to \infty$.

8. **Claim**: If $\max\{c^T x : Ax = b, x \geq 0\}$ is an LP, $\bar{x}$ is feasible, and there is a d with (1) $Ad = \mathbb{O}$, (2) $d \geq \mathbb{O}$, and (3) $c^T d > 0$ then the LP is unbounded.

9. What is a *certificate of unboundedness*?

Certificates of Optimality

10. Argue that $(2,0,0,4,0)^T$ is an optimal solution for the following LP by considering the new system of equations obtained by left-multiplying the constraint-system by $(-1,2)$, rearranging to get 0 and then adding the result to the objective function:

$$\begin{aligned} \max \quad & z(x) = (-1,3,-5,2,1)x - 3 \\ \text{subject to} \quad & \\ & \begin{pmatrix} 1 & -2 & 1 & 0 & 2 \\ 0 & 1 & -1 & 1 & 3 \end{pmatrix} x = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \\ & x \geq \mathbb{0}. \end{aligned} \tag{2.9}$$