Last name \_\_\_\_\_

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## LARSON—MATH 610—HOMEWORK h15 Test 2 Review

**Concepts** For each concept, give a definition and an example.

- 1. What is a generalized eigenvector?
- 2. What is an *invariant subspace* of  $T \in \mathcal{L}(V)$ ?
- 3. What is an *inner product* in a vector space?
- 4. What is the *norm* of a vector v in an inner product space?
- 5. What is the *orthogonal representation* of vectors u, v in an inner product space?
- 6. What is an *orthonormal list* of vectors in an inner product space?
- 7. What is the *orthogonal complement* of a set U in an inner product space V?
- 8. For a subspace U of an inner product space V, what is the orthogonal projection operator  $P_U$ ?
- 9. What is a *linear functional*?
- 10. What is a *self-adjoint* linear operator (on an inner product space)?

Theorems. State (if needed), and provide a proof.

- 11. Suppose  $T \in \mathcal{L}(V)$ . Every list of eigenvectors of T corresponding to distinct eigenvalues is linearly independent.
- 12. What is the Cauchy-Schwartz Inequality in a inner product space?
- 13. What is the *Riesz Representation Theorem*?
- 14. (Claim) Eigenvalues of self-adjoint operators are real.

**Problems** Explain everything. As scientists it is never enough to write answers. They must be communicated—convincingly—to others.

- 15. (Show) For  $T \in \mathcal{L}(V)$  the eigenspace  $U = \{v : T(v) = \lambda v\}$  corresponding to an eigenvalue  $\lambda$  of T is an invariant subspace of T.
- 16. Suppose  $T : \mathbb{C}^3 \to \mathbb{C}^3$ , with  $T(x_1, x_2, x_3) = (4x_1, 0, 5x_2)$ .
  - Find all eigenvalues and associated eigenvectors.
  - Find the eigenspaces corresponding to the eigenvalues of T and check that they do not sum to  $\mathbb{C}^3$ .
  - Find the generalized eigenvectors for  $T: \mathbb{C}^3 \to \mathbb{C}^3$ , with  $T(x_1, x_2, x_3) = (4x_1, 0, 5x_2)$ .
  - For each eigenvalue  $\lambda$  of T find the corresponding set  $G_{\lambda}$  of generalized eigenvectors of T.
  - Show that there is a basis of  $\mathbb{C}^3$  consisting of generalized eigenvectors of T.
  - Show that  $\mathbb{C}^3$  is a direct sum of the generalized eigenspaces corresponding to the eigenvalues of T.
  - Check that the generalized eigenspaces  $G_i$  are invariant under T.
  - For eigenvalues  $\lambda_1, \ldots, \lambda_k$  of T, and generalized eigenspaces  $G_1, \ldots, G_k$ , let  $d_i = \dim G_i$  ( $d_i$  is the *multiplicity* of  $\lambda_i$ ). Check that  $d_1 = \ldots + d_k = \dim(\mathbb{C}^3)$ .
  - Find the characteristic polynomial  $q(x) = (x \lambda_1)^{d_1} \dots (x \lambda_k)^{d_k}$ .
  - Check that q(T) = 0.
- 17. Let V be an inner product space, and  $v \in V$ . Check that  $\langle 0, v \rangle = 0$  and  $\langle v, 0 \rangle = 0$ .
- 18. (Show) An orthonormal list of vectors in an inner product space is linearly independent.
- 19. Describe the *Gram-Schmidt procedure*?
- 20. Use Gram-Schmidt to convert the basis (1,1) and (1,2) in  $\mathbb{R}^2$  into an orthonormal basis.
- 21. (Show:) Every finite-dimensional inner product space has an orthonormal basis.
- 22. (Show) Suppose V is finite-dimensional. Then every orthonormal list of vectors in V can be extended to an orthonormal basis of V.
- 23. (Show:) If U is a subspace of an inner product space V then  $V = U \oplus U^{\perp}$ .
- 24. (Show) Suppose U is a finite-dimensional subspace of  $V, v \in V$ , and  $u \in U$ . Then  $||v P_U v|| \le ||v u||$ .