**Last name** ______________________

**First name** ______________________

# LARSON—MATH 601—HOMEWORK WORKSHEET h15
## The Special Case of Matrices over $\mathbb{R}$

For a matrix $A \in \mathbb{R}^{m\times n}$, let the *transpose* of $A$, $A^t \in \mathbb{R}^{n\times m}$, be the matrix with entries defined by:

$$(A^t)_{i,j} = A_{j,i}.$$

The (square) matrices $A^tA$ and $AA^t$ show up commonly in statistics (variance-covariance matrix), data science, and any time it is useful to compute the very-useful *singular value decomposition*.

1. For any $A \in \mathbb{R}^{m\times n}$, $AA^t$ is symmetric. Explain.

2. It is an immediate corollary of your previous argument that $A^tA$ is symmetric. Why?

3. Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Find $A^t$.

4. Find $AA^t$ and the characteristic values of $AA^t$ (they are real).

5. Find $A^tA$ and the characteristic values of $A^tA$ (they are real).

6. We'll prove the following **claim:** The characteristic values of any symmetric matrix $A \in \mathbb{R}^{n\times n}$ are real.

   The following steps are a **proof**. Your job will be to explain the steps.

   Let $A \in \mathbb{R}^{n\times n}$.

   (a) Why is the characteristic polynomial of $A$, $\det(xI - A)$, guaranteed to have $n$ complex roots?

   Let $c \in \mathbb{C}$, $\alpha \in \mathbb{C}^{n\times 1}$ ($\alpha \neq 0$) be such that $A\alpha = c\alpha$ (such a pair $c, \alpha$ must exist). And let $\bar{\alpha}$ be the $\mathbb{C}^{n\times 1}$ vector whose entries are the complex conjugates of the entries of $\alpha$.

   (b) Argue that $\alpha^t\bar{\alpha}$ is real (or more precisely a $1\times 1$ matrix with a real number entry).

   (c) Let $\bar{A}$ be the matrix who entries are the complex conjugates of the entries of $A$. Explain why $\bar{A} = A$.

Let $\overline{A\alpha}$ be the matrix who entries are the complex conjugates of the entries of $A\alpha$.

(d) Explain why $\overline{A\alpha} = \bar{A}\bar{\alpha}$. (And thus $\overline{A\alpha} = A\bar{\alpha}$).

Let $\overline{c\alpha}$ be the matrix who entries are the complex conjugates of the entries of $c\alpha$.

(e) Explain why $\overline{c\alpha} = \bar{c}\bar{\alpha}$.

So, $\overline{A\alpha} = \overline{c\alpha}$ implies $A\bar{\alpha} = \bar{c}\bar{\alpha}$, and $\alpha^t A\bar{\alpha} = \alpha^t\bar{c}\bar{\alpha} = \bar{c}\alpha^t\bar{\alpha}$.

Also, $A\alpha = c\alpha$ implies $(A\alpha)^t = (c\alpha)^t$, which implies $\alpha^t A = c\alpha^t$, and thus $\alpha^t A\bar{\alpha} = c\alpha^t\bar{\alpha}$.

(f) Explain why $\bar{c}\alpha^t\bar{\alpha} = c\alpha^t\bar{\alpha}$.

(g) Explain why $\bar{c} = c$.

(h) Explain why $c$ must be a real number (and thus every characteristic value of $A$ is real).