Last name ________________________

First name ________________________

LARSON—MATH 610–HOMEWORK WORKSHEET 13
Test 2 Review.

Definitions. Write each definition **and give an example**.

1. What is an *eigenvalue* and *eigenvector* of a matrix $A \in \mathbb{M}_n(\mathbb{F})$.
2. What is a *unitary* matrix?
3. When are matrices *unitarily similar*?
4. What is the *spectrum* of a square matrix?
5. For $A = [a_{ij}] \in \mathbb{M}_n$ ($n \geq 2$) what is the *Gershgorin disk* $G_k(A)$?
6. diagonally dominant square matrix
7. What is a *normal* matrix?
8. What is a *positive definite* matrix?
9. What is a *positive semi-definite* matrix?
10. What is a *Rayleigh quotient*?

Notation. Write each definition **and give an example**.

11. Let $\mathcal{V}$ be an inner product space and $\mathcal{U}$ be a subspace. What is $\mathcal{U}^{\perp}$?

Theorems. State each theorem carefully, and **give an example**.

12. Gershgorin's Disc Theorem.
13. Schur Triangularization.
14. Spectral Theorem.
15. Rayleigh's Theorem.
16. Cauchy's Interlacing Theorem.

Algorithms. State each algorithm and **give an example**.

17. Gram-Schmidt.

Problems. Carefully **explain** your answer.

18. Prove: a real symmetric matrix has only real eigenvalues.

19. Let $\mathcal{V}$ be a finite-dimensional inner product space, let $\beta = \hat{u}_1, \ldots, \hat{u}_n$ be an orthonormal basis for $\mathcal{V}$ and let $\hat{v} \in \mathcal{V}$. Let $\hat{v} = c_1\hat{u}_1 + \ldots c_n\hat{u}_n$. Find c_i.

20. Let $\mathcal{V}$ be a finite-dimensional inner product space, let $\beta = \hat{u}_1, \ldots, \hat{u}_n$ be an orthonormal basis for $\mathcal{V}$ and let $\hat{v} \in \mathcal{V}$. Let $\hat{v} = c_1\hat{u}_1 + \ldots c_n\hat{u}_n$. Show: $\sum_{i=1}^{n} |c_i|^2 = 1$.

21. Argue: Every finite-dimensional inner product space has an orthonormal basis.

22. Argue: An orthogonal list of vectors in an inner product space is linearly independent.

23. Let $U \in \mathbb{M}_n(\mathbb{F})$. Argue: if U is unitary then $||\hat{x}||_2 = ||U\hat{x}||_2$ for all $\hat{x} \in \mathbb{F}^n$.

24. Let $\mathcal{V}$ be an inner product space. Argue: If $\mathcal{U}$ is a finite-dimensional subspace of $\mathcal{V}$ then $\mathcal{V} = \mathcal{U} \oplus \mathcal{U}^{\perp}$.

25. Show: the numbers on the diagonal of a diagonal matrix $A \in \mathbb{M}_m(\mathtt{F})$ are eigenvalues.

26. Show: if A and B are similar then they have the same eigenvalues.

27. . Let $(\lambda_1, \hat{x}_1), \ldots, (\lambda_d, \hat{x}_d)$ be eigenpairs of $A \in \mathbb{M}_n$, in which $\lambda_1, \ldots, \lambda_d$ are all distinct. Show: $\hat{x}_1, \ldots, \hat{x}_d$ are linearly independent.

28. What can we say about the spectrum of $A = \begin{bmatrix} 1 & .1 & .2 \\ .3 & 2 & .3 \\ .4 & .5 & 7 \end{bmatrix}$?

29. Show: If A is strictly diagonally dominant, then it is invertible.

30. Let $\hat{x}$ be any unit vector in $\mathbb{R}^3$. How can you find a unitary matrix U whose first column in $\hat{x}$?

31. (**Show:**) If a matrix $T \in \mathbb{M}_2(\mathbb{C})$ is normal then is is diagonal.

32. Let $A \in \mathbb{M}_{m \times n}(\mathbb{R})$. Show: $A^T A$ a positive semi-definite matrix.

33. How can Rayleigh quotients be used to bound the largest and smallest eigenvalues of a Hermitian matrix?