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First name _____

LARSON—MATH 610—CLASSROOM WORKSHEET 37
Positive Definite matrices, Rayleigh quotients.

(Chp. 5). *inner product, inner product space, $\langle \cdot, \cdot \rangle$, orthogonal vectors, \perp , $\|\cdot\|$.*

(Chp. 6). *orthogonal basis.*

(Chp. 7). *unitary matrix, unitarily similar.*

(Chp. 8). *U^\perp , $V = U \oplus U^\perp$, orthonormal projection, $P_U \hat{v}$.*

(Chp. 9). *eigenvalue, eigenvector, eigenpair, nullity, annihilating polynomial, spectrum, Gershgorin disk, diagonally dominant. (Chp. 15). positive semi-definite, square root $A^{\frac{1}{2}}$.*

(Chp. 18). *Rayleigh quotient.*

Review:

1. (**Theorem 14.2.2. Spectral Theorem**). Let $A \in \mathbb{M}_n$, let $\lambda_1, \dots, \lambda_n$ be its eigenvalues (in any order) and let $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$. The following are equivalent.

(a) A is normal (that is, $AA^* = A^*A$).

(b) A is unitarily diagonalizable: there is a unitary matrix $U \in \mathbb{M}_n$ such that

$$A = U\Lambda U^*.$$

(c) There is an orthonormal basis $\hat{u}_1, \dots, \hat{u}_n$ of \mathbb{C}^n such that

$$A = \lambda_1 \hat{u}_1 \hat{u}_1^* + \dots + \lambda_n \hat{u}_n \hat{u}_n^*.$$

(d) \mathbb{C}^n has an orthonormal basis consisting of eigenvectors of A .

2. What is a *positive definite* matrix?

3. What is a *positive semi-definite* (PSD) matrix?

Chp. 15 of Garcia & Horn, Matrix Mathematics

1. Let $A \in \mathbb{M}_{m \times n}(\mathbb{R})$. Why is $A^T A$ a PSD matrix?

Chp. 15 of Garcia & Horn, Matrix Mathematics

2. (**Theorem 15.1.3**). Let $A \in \mathbb{M}_n(\mathbb{F})$. The following are equivalent.

(a) A is positive semi-definite.

(b) A is Hermitian and all its eigenvalues are non-negative.

(c) There is a $B \in \mathbb{M}_n(\mathbb{F})$ such that $A = B^* B$.

3. Why is this theorem true?

Chp. 18 of Garcia & Horn, Matrix Mathematics

4. What is a *Rayleigh quotient*?

5. How can Rayleigh quotients be used to bound the largest and smallest eigenvalues of a Hermitian matrix?