Last name ______________________

First name ______________________

LARSON—MATH 610–CLASSROOM WORKSHEET 36
Spectral Theorem, Positive Definite matrices.

(Chp. 5). *inner product*, *inner product space*, $\langle\ \rangle$, *orthogonal* vectors, $\perp$, $||\cdot||$.
(Chp. 6). *orthogonal basis.*
(Chp. 7). *unitary matrix*, *unitarily similar.*
(Chp. 8). $U^\perp$, $V = U \oplus U^\perp$, *orthonormal projection*, $P_{\mathcal{U}}\hat{v}$.
(Chp. 9). eigenvalue, eigenvector, eigenpair, nullity, annihilating polynomial, spectrum, Gershgorin disk, *diagonally dominant.* **(Chp. 15)**. positive semi-definite, square root $A^{\frac{1}{2}}$.

Review:

1. (**Claim:**) If a triangular matrix $T \in \mathbb{M}_n(\mathbb{C})$ is normal then is is diagonal.

Chp. 14 of Garcia & Horn, Matrix Mathematics

(**Theorem 14.2.2. Spectral Theorem**). Let $A \in \mathbb{M}_n$), let $\lambda_1, \ldots, \lambda_n$ be its eigenvalues (in any order) and let $\Lambda = diag(\lambda_1, \ldots, \lambda_n)$. The following are equivalent.

(a) A is normal (that is, $AA^* = A^*A$).

(b) A is unitarily diagonalizable: there is a unitary matrix $U \in \mathbb{M}_n$ such that

$$A = U\Lambda U^*.$$

(c) There is an orthonormal basis $\hat{u}_1, \ldots, \hat{u}_n$ of $\mathbb{C}^n$ such that

$$A = \lambda_1\hat{u}_1\hat{u}_1^* + \ldots + \lambda_n\hat{u}_n\hat{u}_n^*.$$

(d) $\mathbb{C}^n$ has an orthonormal basis consisting of eigenvectors of A.

1. Why is this theorem true?

2. What is an example?

3. What is the importance of *normality*?

4. What is a *positive definite* matrix?

5. What is an example?

6. What is a *positive semi-definite* matrix?

7. What is an example?

8. (**Theorem 15.1.3**). Let $A \in \mathbb{M}_n(\mathbb{F})$. The following are equivalent.
   (a) A is positive semi-definite.
   (b) A is Hermitian and all its eigenvalues are non-negative.
   (c) There is a $B \in \mathbb{M}_n(\mathbb{F})$ such that $A = B^*B$.
   (d) For some positive integer m, there is a $B \in \mathbb{M}_{m\times n}(\mathbb{F})$ such that $A = B^*B$.

9. Why is this theorem true?