Last name ______________________

First name ______________________

LARSON—MATH 610–CLASSROOM WORKSHEET 34
Schur Triangularization.

(Chp. 5). *inner product*, *inner product space*, $\langle\ \rangle$, *orthogonal* vectors, $\perp$, $||\cdot||$.
(Chp. 6). *orthogonal basis*.
(Chp. 7). *unitary matrix*, *unitarily similar*.
(Chp. 8). $U^\perp$, $V = U \oplus U^\perp$, *orthonormal projection*, $P_{\mathcal{U}}\hat{v}$.
(Chp. 9). eigenvalue, eigenvector, eigenpair, nullity, annihilating polynomial, spectrum, Gershgorin disk, *diagonally dominant*.

Review:

1. (**Theorem 11.1 Schur Triangularization**) Let the eigenvalues of $A \in \mathbb{M}_n$ be arranged in any order $\lambda_1, \ldots, \lambda_n$ (with any repetitions) and let $(\lambda_i, \hat{x})$ be an eigenpair of A, in which $\hat{x}$ is a unit vector. Then,

   (a) there is a unitary $U \in \mathbb{M}_n$ with column $\hat{x}$ such that $A = UTU^*$, in which $T = [t_{ij}]$ is upper-triangular and has diagonal entries $t_{ii} = \lambda_i$, for $i = 1, \ldots, n$.

Chp. 11 of Garcia & Horn, Matrix Mathematics

(b) If A is real, $\lambda_1, \ldots, \lambda_n$ are real, and $\hat{x}$ is real, then there is a real orthogonal matrix $Q \in \mathbb{M}_n(\mathbb{R})$ with first column $\hat{x}$ such that $A = QTQ^T$ in which $T = [t_{ij}]$ is real upper-triangular and has diagonal entries $t_{ii} = \lambda_i$, for $i = 1, \ldots, n$.

1. Why is this theorem true?

Chp. 14 of Garcia & Horn, Matrix Mathematics

2. What is a *normal* matrix?

3. (**Claim:**) If a matrix $T \in \mathbb{M}_2(\mathbb{C})$ is normal then is is diagonal.

4. (**Claim:**) If a matrix $T \in \mathbb{M}_n(\mathbb{C})$ is normal then is is diagonal.

(**Theorem 14.2.2. Spectral Theorem**). Let $A \in \mathbb{M}_n$, let $\lambda_1, \ldots, \lambda_n$ be its eigenvalues (in any order) and let $\Lambda = diag(\lambda_1, \ldots, \lambda_n)$. The following are equivalent.

(a) A is normal (that is, $AA^* = A^*A$).

(b) A is unitarily diagonalizable: there is a unitary matrix $U \in \mathbb{M}_n$ such that

$$A = U\Lambda U^*.$$

(c) There is an orthonormal basis $\hat{u}_1, \ldots, \hat{u}_n$ of $\mathbb{C}^n$ such that

$$A = \lambda_1 \hat{u}_1 \hat{u}_1^* + \ldots + \lambda_n \hat{u}_n \hat{u}_n^*.$$

(d) $\mathbb{C}^n$ has an orthonormal basis consisting of eigenvectors of A.

5. Why is this theorem true?

6. What is an example?

7. What is the importance of *normality*?