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LARSON—MATH 610—CLASSROOM WORKSHEET 33
Schur Triangularization.

(Chp. 5). *inner product, inner product space, $\langle \cdot, \cdot \rangle$, orthogonal vectors, \perp , $\|\cdot\|$.*

(Chp. 6). *orthogonal basis.*

(Chp. 7). *unitary matrix, unitarily similar.*

(Chp. 8). *U^\perp , $V = U \oplus U^\perp$, orthonormal projection, $P_U \hat{v}$.*

(Chp. 9). *eigenvalue, eigenvector, eigenpair, nullity, annihilating polynomial, spectrum, Gershgorin disk, diagonally dominant.*

Review:

1. The diagonal entries of a triangular matrix $A \in \mathbb{M}_n(\mathbb{C})$ are eigenvalues.
2. Let \hat{x} be any unit vector in \mathbb{R}^3 . How can you find a unitary matrix U whose first column in \hat{x} ?
3. Let \hat{x} be any unit vector in \mathbb{C}^n . How can you find a unitary matrix U whose first column in \hat{x} ?

Chp. 11 of Garcia & Horn, Matrix Mathematics

(**Theorem 11.1 Schur Triangularization**) Let the eigenvalues of $A \in \mathbb{M}_n$ be arranged in any order $\lambda_1, \dots, \lambda_n$ (with any repetitions) and let (λ_i, \hat{x}) be an eigenpair of A , in which \hat{x} is a unit vector. Then,

- (a) there is a unitary $U \in \mathbb{M}_n$ with column \hat{x} such that $A = UTU^*$, in which $T = [t_{ij}]$ is upper-triangular and has diagonal entries $t_{ii} = \lambda_i$, for $i = 1, \dots, n$.

1. Why is this theorem true?

- (b) If A is real, $\lambda_1, \dots, \lambda_n$ are real, and \hat{x} is real, then there is a real orthogonal matrix $Q \in \mathbb{M}_n(\mathbb{R})$ with first column \hat{x} such that $A = QTQ^T$ in which $T = [t_{ij}]$ is real upper-triangular and has diagonal entries $t_{ii} = \lambda_i$, for $i = 1, \dots, n$.

2. Why is this theorem true?

Chp. 14 of Garcia & Horn, Matrix Mathematics

(Theorem 14.2.2. Spectral Theorem). Let $A \in \mathbb{M}_n$, let $\lambda_1, \dots, \lambda_n$ be its eigenvalues (in any order) and let $A = \text{diag}(\lambda_1, \dots, \lambda_n)$. The following are equivalent.

- (a) A is normal (that is, $AA^* = A^*A$).
(b) A is unitarily diagonalizable: there is a unitary matrix $U \in \mathbb{M}_n$ such that

$$A = U\Lambda U^*.$$

- (c) There is an orthonormal basis $\hat{u}_1, \dots, \hat{u}_n$ of \mathbb{C}^n such that

$$A = \lambda_1 \hat{u}_1 \hat{u}_1^* + \dots + \lambda_n \hat{u}_n \hat{u}_n^*.$$

- (d) \mathbb{C}^n has an orthonormal basis consisting of eigenvectors of A .

3. Why is this theorem true?

4. What is an example?

5. What is the importance of *normality*?