Last name ______________________

First name ______________________

LARSON—MATH 610–CLASSROOM WORKSHEET 31
Schur Triangularization.

(Chp. 5). *inner product*, *inner product space*, $\langle\ \rangle$, *orthogonal* vectors, $\perp$, $||\cdot||$.
(Chp. 6). *orthogonal basis*.
(Chp. 7). *unitary matrix*, *unitarily similar*.
(Chp. 8). $U^{\perp}$, $V = U \oplus U^{\perp}$, *orthonormal projection*, $P_{\mathcal{U}}\hat{v}$.
(Chp. 9). eigenvalue, eigenvector, eigenpair, nullity, annihilating polynomial, spectrum, Gershgorin disk, *diagonally dominant*.

Review:

1. (**Corollary 9.4.15**). Let $A \in \mathcal{M}_n$.
   (a) If A is strictly diagonally dominant, then it is invertible.
   (b) Suppose that A has all real eigenvalues and real non-negative diagonal entries. If A is diagonally dominant, then all of its eigenvalues are non-negative. If A is strictly diagonally dominant, then all its eigenvalues are positive.

1. We showed that a matrix $A \in \mathbb{C}^n$ has a 0 eigenvalue if and only if $nullity(A) > 0$. Why is any basis vecor of the null space an eigenvector?

Chp. 11 of Garcia & Horn, Matrix Mathematics

2. Let $\hat{x}$ be any unit vector in $\mathbb{R}^3$. How can you find a unitary matrix U whose first column in $\hat{x}$?

3. Let $\hat{x}$ be any unit vector in $\mathbb{C}^n$. How can you find a unitary matrix U whose first column in $\hat{x}$?

4. Argue that the diagonal entries of any triangular matrix $A \in \mathbb{M}_2(\mathbb{C})$ are eigenvalues.

5. Argue that any matrix $A \in \mathbb{M}_2(\mathbb{C})$ can be written $A = UTU^*$, where U is unitary and T is triangular.

6. (**Theorem 11.1 Schur Triangularization**) Let the eigenvalues of $A \in \mathbb{M}_n$ be arranged in any order $\lambda_1, \ldots, \lambda_n$ (with any repetitions) and let $(\lambda_i, \hat{x})$ be an eigenpair of A, in which $\hat{x}$ is a unit vector. Then,

   (a) there is a unitary $U \in \mathbb{M}_n$ with column $\hat{x}$ such that $A = UTU^*$, in which $T = [t_{ij}]$ is upper-triangular and has diagonal entries $t_{ij} = \lambda_i$, for $i = 1, \ldots, n$.

7. Why is this theorem true?

   (b) If A is real, $\lambda_1, \ldots, \lambda_n$ are real, and $\hat{x}$ is real, then there is a real orthogonal matrix $Q \in \mathbb{M}_n(\mathbb{R})$ with first column $\hat{x}$ such that $A = QTQ^T$ in which $T = [t_{ij}]$ is real upper-triangular and has diagonal entries $t_{ij} = \lambda_i$, for $i = 1, \ldots, n$.

8. Why is this theorem true?