Last name ______________________

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LARSON—MATH 610–CLASSROOM WORKSHEET 30
Eigenvalues, Gershgorin Discs.

(Chp. 5). *inner product*, *inner product space*, $\langle\ \rangle$, *orthogonal* vectors, $\perp$, $||\cdot||$.
(Chp. 6). *orthogonal basis*.
(Chp. 7). *unitary matrix*, *unitarily similar*.
(Chp. 8). $U^{\perp}$, $V = U \oplus U^{\perp}$, *orthonormal projection*, $P_{\mathcal{U}}\hat{v}$.
(Chp. 9). eigenvalue, eigenvector, eigenpair, nullity, annihilating polynomial, spectrum, Gershgorin disk.

Review:

1. (**Theorem 9.3.5.**). Let $(\lambda_1, \hat{x}_1), \ldots, (\lambda_d, \hat{x}_d)$ be eigenpairs of $A \in \mathbb{M}_n$, in which $\lambda_1, \ldots, \lambda_d$ are all distinct. Then $\hat{x}_1, \ldots, \hat{x}_d$ are linearly independent.
2. What is the *spectrum* of a square matrix?
3. For $A = [a_{ij}] \in \mathbb{M}_n$ $(n \geq 2)$ what is the *Gershgorin disk* $G_k(A)$?

(**Theorem 9.4.2. Gershgorin disk**). If $n \geq 2$ and $A = [a_{ij}] \in \mathbb{M}_n$, then

$$spec(A) \subseteq \mathcal{G}(A) = \cup_{k=1}^{n} \mathcal{G}_k(A).$$

1. Why is this theorem true?

2. (**Corollary 9.4.15**). Let $A \in \mathcal{M}_n$.
   (a) If A is strictly diagonally dominant, then it is invertible.
   (b) Suppose that A has all real eigenvalues and real non-negative diagonal entries. If A is diagonally dominant, then all of its eigenvalues are non-negative. If A is strictly diagonally dominant, then all its eigenvalues are positive.

3. What can we say about $A = \begin{bmatrix} 1 & .1 & .2 \\ .3 & 2 & .3 \\ .4 & .5 & 7 \end{bmatrix}$?

Chp. 11 of Garcia & Horn, Matrix Mathematics

4. (**Theorem 11.1 Schur Triangularization**) Let the eigenvalues of $A \in \mathbb{M}_n$ be arranged in any order $\lambda_1, \ldots, \lambda_n$ (with any repetitions) and let $(\lambda_i, \hat{x})$ be an eigenpair of A, in which $\hat{x}$ is a unit vector. Then,

   (a) there is a unitary $U \in \mathbb{M}_n$ with column $\hat{x}$ such that $A = UTU^*$, in which $T = [t_{ij}]$ is upper-triangular and has diagonal entries $t_{ij} = \lambda_i$, for $i = 1, \ldots, n$.

5. Why is this theorem true?

   (b) If A is real, $\lambda_1, \ldots, \lambda_n$ are real, and $\hat{x}$ is real, then there is a real orthogonal matrix $Q \in \mathbb{M}_n(\mathbb{R})$ with first column $\hat{x}$ such that $A = QTQ^T$ in which $T = [t_{ij}]$ is real upper-triangular and has diagonal entries $t_{ij} = \lambda_i$, for $i = 1, \ldots, n$.

6. Why is this theorem true?