Last name ______________________

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LARSON—MATH 610—CLASSROOM WORKSHEET 17
Generalized Eigenvectors.

Concepts & Notation

- (Chp. 5) *eigenvalue*, *eigenvector*, *invariant subspace*, *minimal polynomial*,
- (Chp. 8) *generalized eigenvector.*

Complex Vector Spaces

Let V be a finite-dimensional complex vector space and $T \in \mathcal{L}(V)$, with eigenvalues $\lambda_1, \ldots, \lambda_k$, generalized eigenspaces $G_1, \ldots, G_k$, with $d_i = dim\ G_i$.

1. Generalized eigenvectors corresponding to distinct eigenvalues are linearly independent.
2. There is a basis of V consisting of generalized eigenvectors of T.
3. $V = G_1 \oplus \ldots \oplus G_k$.
4. $d_1 + \ldots + d_k = dim(V)$.
5. (Def.) The **characteristic polynomial** of T is $q(x) = (x - \lambda_1)^{d_1} \ldots (x - \lambda_k)^{d_k}$.
6. (**Cayley-Hamilton Theorem**). $q(T) = 0$.
7. The minimal polynomial of T divides the characteristic polynomial of T.

Our Example

1. Suppose $T : \mathbb{C}^3 \to \mathbb{C}^3$, with $T(x_1, x_2, x_3) = (4x_1, 0, 5x_2)$. Find all eigenvalues and associated eigenvectors.

2. Find the eigenspaces corresponding to the eigenvalues of T and check that they do not sum to $\mathbb{C}^3$.

3. What is a *generalized eigenvector*?

4. Find the generalized eigenvectors for $T : \mathbb{C}^3 \to \mathbb{C}^3$, with $T(x_1, x_2, x_3) = (4x_1, 0, 5x_2)$.

5. For each eigenvalue λ of T find the corresponding set G_λ of generalized eigenvectors of T.

6. Show that there is a basis of $\mathbb{C}^3$ consisting of generalized eigenvectors of T.
7. Show that $\mathbb{C}^3$ is a direct sum of the generalized eigenspaces corresponding to the eigenvalues of T.
8. Check that the generalized eigenspaces G_i are invariant under T.
9. For eigenvalues $\lambda_1, \ldots, \lambda_k$ of T, and generalized eigenspaces $G_1, \ldots, G_k$, let $d_i = dim\ G_i$ (d_i is the *multiplicity* of λ_i). Check that $d_1 = \ldots + d_k = dim(\mathbb{C}^3)$.
10. Find the characteristic polynomial $q(x) = (x - \lambda_1)^{d_1} \ldots (x - \lambda_k)^{d_k}$.
11. Check that $q(T) = 0$.

12. Explain why the minimal polynomial $p(x)$ of T must divide the characteristic polynomial $q(x)$ of T. Use this fact and other facts we proved to make conclusions about the minimal polynomial of T.

Inner Product Spaces

13. What is the *dot product* of vectors in $\mathbb{R}^n$?

14. What is an *inner product* in a vector space? Check that the dot product in $\mathbb{R}^2$ is an inner product.