Last name ______________________

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LARSON—MATH 610—CLASSROOM WORKSHEET 15
Invariant Subspaces, Upper-Triangular Matrices, and Generalized Eigenvectors.

Concepts & Notation

- (Chp. 5) *eigenvalue*, *eigenvector*, *invariant subspace*, *minimal polynomial*,
- (Chp. 8) *generalized eigenvector*.

Minimal Polynomials

1. (**Existence, uniqueness, and degree of minimal polynomial**). If V is a finite-dimensional vector space, and $T \in \mathcal{L}(V)$, then there is a unique monic polynomial $p \in \mathcal{P}(\mathbb{F})$ of smallest degree with $p(T) = 0$.

2. Suppose V is a finite-dimensional complex vector space, $T \in \mathcal{L}(V)$ and $p(T)$ is the minimal polynomial. Then the roots of p are exactly the eigenvalues of T.

Invariant Subspaces and Upper-Triangular Matrices

3. What is an *invariant subspace* of $T \in \mathcal{L}(V)$?

4. (**Claim**:) Suppose $T \in \mathcal{L}(V)$ and $(v_1, \ldots, v_n)$ is a basis of V. Then the following are equivalent:

   (a) the matrix of T with respect to $(v_1, \ldots, v_n)$ is upper-triangular;

   (b) $T(v_k) \in span(v_1, \ldots, v_k)$ for each $k = 1, \ldots, n$;

   (c) $span(v_1, \ldots, v_k)$ is invariant under T for each $k = 1, \ldots, n$.

5. (**Claim**:) Suppose V is a finite-dimensional complex vector space and $T \in \mathcal{L}(V)$. Then T has an upper-triangular matrix with respect to some basis of V.

Complex Vector Spaces

6. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$, with $T(x_1, x_2, x_3) = (4x_1, 0, 5x_2)$. Find all eigenvalues and associated eigenvectors.

7. What is a *generalized eigenvector*?

8. Find the generalized eigenvectors for $T : \mathbb{R}^3 \to \mathbb{R}^3$, with $T(x_1, x_2, x_3) = (4x_1, 0, 5x_2)$.