Last name ______________________

First name ______________________

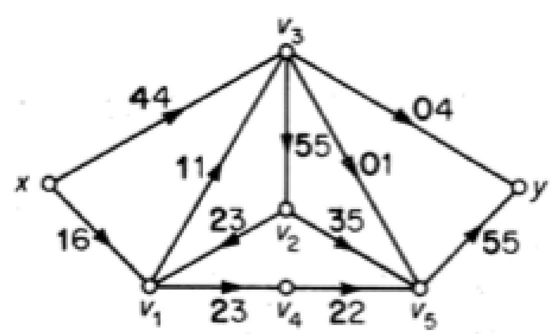
LARSON—MATH 556—CLASSROOM WORKSHEET 25
Max Flow-Min Cut Theorem

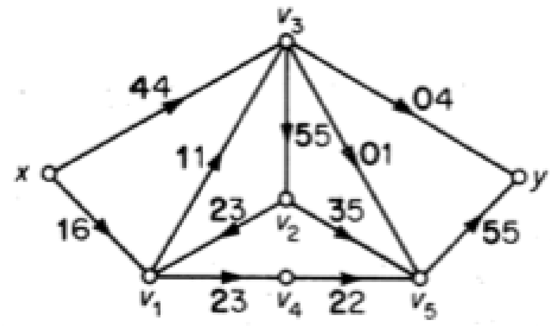
Review

- What is a *directed graph*?
- What is a *source* in a directed graph?
- What is a *sink* in a directed graph?
- What is a *capacity* of a line in a directed graph?
- What is a *network*?
- What is a *flow* in a network?
- What is the *value* of a flow in a network?
- Why does a maximum flow in a network *exist*?
- If $A \subseteq V(D)$, what is the *directed cut out of* A, $\nabla^+(A)$?
- What is a *separator* A in a network?
- What is the *capacity* of a cut $\nabla^+(A)$ (or a separator A) in a network?

Network Flows

1. Here the source is x and the sink is y. On each line, the flow is listed first, followed by the capacity. Check that the flow is valid.





2. Find $val(f)$.

3. Let $A = \{x, v_1, v_2\}$. Find $\nabla^+(A)$ and $cap(\nabla^+(A))$.

4. Check that $val(f) \leq cap(\nabla^+(A))$.

5. We could alternatively define the value of a flow to be the net flow into the sink. Check that this comes out the same. What is the corresponding separator and cut?

6. Why is the value of a flow in a network no more than the capacity of any cut?

7. Explain the following proof.

2.1.2. LEMMA. *If f is any flow in D and C is any $s-t$ cut, then* $\text{val}(f) \leq \text{cap}(C)$.

PROOF. Let f and $C = \nabla^+(A)$ denote an arbitrary $s-t$ flow and an $s-t$ cut in D respectively. Then

$$\begin{aligned}
\text{val}(f) &= \sum_u f(s,u) - \sum_u f(u,s) \\
&= \sum_u f(s,u) - \sum_u f(u,s) + \sum_{a\in A-s} \Big(\sum_w f(a,w) - \sum_v f(v,a)\Big) \\
&= \sum_{a\in A} \Big(\sum_w f(a,w) - \sum_v f(v,a)\Big) \\
&= \sum_{a\in A} \sum_w f(a,w) - \sum_{a\in A} \sum_v f(v,a) \\
&= \Big(\sum_{\substack{a\in A\\ w\in A}} f(a,w) + \sum_{\substack{a\in A\\ w\in V-A}} f(a,w)\Big) - \Big(\sum_{\substack{a\in A\\ v\in A}} f(v,a) + \sum_{\substack{a\in A\\ v\in V-A}} f(v,a)\Big)
\end{aligned}$$

Noting that the first and third terms cancel we have

$$\text{val}(f) = \sum_{\substack{a\in A\\ w\in V-A}} f(a,w) - \sum_{\substack{a\in A\\ v\in V-A}} f(v,a).$$

But by definition of flow, $\sum_{a\in A, v\in V-A} f(v,a) \geq 0$, so

$$\text{val}(f) \leq \sum_{\substack{a\in A\\ w\in V-A}} f(a,w) \leq \sum_{\substack{a\in A\\ w\in V-A}} c(a,w) \leq \text{cap}(A).$$

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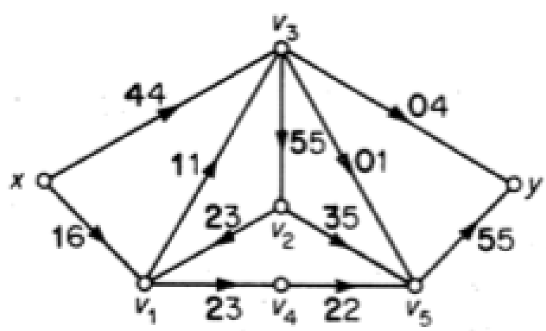

8. What is an *f-augmenting path to u_k* in a network? What is an *f-augmenting path* in a network?

9. Does this network have a flow-augmenting path?

10. (**Claim:**) A flow f is maximum if and only if there are no f-augmenting paths.

11. (**Max-Flow Min-Cut Theorem:**) The value of a maximum flow in a network equals the capacity of a minimum cut.