Last name _____

First name _____

LARSON—MATH 556—CLASSROOM WORKSHEET 24
Max Flow-Min Cut Theorem

Review

- What is a *directed graph*?
- What is a *source* in a directed graph?
- What is a *sink* in a directed graph?
- What is a *capacity* of a line in a directed graph?
- What is a *network*?

Network Flows

1. What is a *flow* in a network?

2. What is the *value* of a flow in a network?

3. Why does a maximum flow in a network *exist*?

4. If $A \subseteq V(D)$, what is the *directed cut out of* A, $\nabla^+(A)$?

5. What is a *separator* A in a network?

6. What is the *capacity* of a cut $\nabla^+(A)$ (or a separator A) in a network?

7. What is the value of a flow in a network no more than the capacity of any cut?

8. Explain the following proof.

2.1.2. LEMMA. *If f is any flow in D and C is any $s-t$ cut, then* $\mathrm{val}(f) \le \mathrm{cap}(C)$.

PROOF. Let f and $C = \nabla^+(A)$ denote an arbitrary $s-t$ flow and an $s-t$ cut in D respectively. Then

$$\begin{aligned}
\mathrm{val}(f) &= \sum_u f(s,u) - \sum_u f(u,s) \\
&= \sum_u f(s,u) - \sum_u f(u,s) + \sum_{a \in A-s} \Big(\sum_w f(a,w) - \sum_v f(v,a) \Big) \\
&= \sum_{a \in A} \Big(\sum_w f(a,w) - \sum_v f(v,a) \Big) \\
&= \sum_{a \in A} \sum_w f(a,w) - \sum_{a \in A} \sum_v f(v,a) \\
&= \Big(\sum_{\substack{a \in A \\ w \in A}} f(a,w) + \sum_{\substack{a \in A \\ w \in V-A}} f(a,w) \Big) - \Big(\sum_{\substack{a \in A \\ v \in A}} f(v,a) + \sum_{\substack{a \in A \\ v \in V-A}} f(v,a) \Big)
\end{aligned}$$

Noting that the first and third terms cancel we have

$$\mathrm{val}(f) = \sum_{\substack{a \in A \\ w \in V-A}} f(a,w) - \sum_{\substack{a \in A \\ v \in V-A}} f(v,a).$$

But by definition of flow, $\sum_{a \in A, v \in V-A} f(v,a) \ge 0$, so

$$\mathrm{val}(f) \le \sum_{\substack{a \in A \\ w \in V-A}} f(a,w) \le \sum_{\substack{a \in A \\ w \in V-A}} c(a,w) \le \mathrm{cap}(A).$$

■

9. What is an *f-augmenting path to u_k* in a network?

10. What is an *f-augmenting path* in a newtwork?

11. (**Claim:**) A flow f is maximum if and only if there are no f-augmenting paths.