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## LARSON—MATH 556—CLASSROOM WORKSHEET 24 Max Flow-Min Cut Theorem

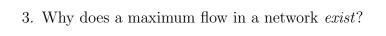
## Review

- What is a directed graph?
- What is a *source* in a directed graph?
- What is a *sink* in a directed graph?
- What is a *capacity* of a line in a directed graph?
- What is a *network*?

## **Network Flows**

1. What is a *flow* in a network?

2. What is the *value* of a flow in a network?



4. If 
$$A \subseteq V(D)$$
, what is the directed cut out of  $A, \nabla^+(A)$ ?

5. What is a *separator* 
$$A$$
 in a network?

6. What is the *capacity* of a cut 
$$\nabla^+(A)$$
 (or a separator A) in a network?

7. What is the value of a flow in a network no more than the capacity of any cut?

8. Explain the following proof.

**2.1.2. LEMMA.** If f is any flow in D and C is any s-t cut, then  $val(f) \le cap(C)$ .

**PROOF.** Let f and  $C = \nabla^+(A)$  denote an arbitrary s - t flow and an s - t cut in D respectively. Then

$$\begin{aligned} \operatorname{val}(f) &= \sum_{u} f(s, u) - \sum_{u} f(u, s) \\ &= \sum_{u} f(s, u) - \sum_{u} f(u, s) + \sum_{a \in A - s} \left( \sum_{w} f(a, w) - \sum_{v} f(v, a) \right) \\ &= \sum_{a \in A} \left( \sum_{w} f(a, w) - \sum_{v} f(v, a) \right) \\ &= \sum_{a \in A} \sum_{w} f(a, w) - \sum_{a \in A} \sum_{v} f(v, a) \\ &= \left( \sum_{\substack{a \in A \\ w \in A}} f(a, w) + \sum_{\substack{a \in A \\ w \in V - A}} f(a, w) \right) - \left( \sum_{\substack{a \in A \\ v \in A}} f(v, a) + \sum_{\substack{a \in A \\ v \in V - A}} f(v, a) \right) \end{aligned}$$

Noting that the first and third terms cancel we have

$$\operatorname{val}(f) = \sum_{\substack{a \in A \\ w \in V - A}} f(a, w) - \sum_{\substack{a \in A \\ v \in V - A}} f(v, a).$$

But by definition of flow,  $\sum_{a \in A, v \in V-A} f(v, a) \ge 0$ , so

$$\operatorname{val}(f) \leq \sum_{\substack{a \in A \\ w \in V - A}} f(a, w) \leq \sum_{\substack{a \in A \\ w \in V - A}} c(a, w) \leq \operatorname{cap}(A).$$

9. What is an $f$ -augmenting path to $u_k$ in a network?
10. What is an $f$ -augmenting path in a newtwork?
11. (Claim:) A flow $f$ is maximum if and only if there are no $f$ -augmenting paths.