Last name ______________________

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LARSON—MATH 556—CLASSROOM WORKSHEET 23
Max Flow-Min Cut Theorem

Review

- What is the *maximum degree* Δ of a graph?
- What is a *regular* graph?
- What is a *valid* (or *proper*) *line coloring* of a graph? What is χ_e?
- What is Kőnig's Line Coloring Theorem?
- Why does a regular bipartite graph have a perfect matching?
- Lovasz and Plummer claim that, given any bipartite graph G with maximum degree Δ there is a Δ-regular bipartite graph H where G is a subgraph of H. Why is that true?

Questions

1. Prove Kőnig's Line Coloring Theorem.

Network Flows

2. What is a *directed graph*?

3. What is a *source* in a directed graph?

4. What is a *sink* in a directed graph?

5. What is a *capacity* of a line in a directed graph?

6. What is a *network*?

7. What is a *flow* in a network?

8. What is the *value* of a flow in a network?

9. Why does a maximum flow in a network *exist*?

10. If $A \subseteq V(D)$, what is the *directed cut out of* A, $\nabla^+(A)$?

11. What is a *separator* A in a network?

12. What is a *cut* $\nabla(A)^+$ in a network?

13. What is the *capacity* of a cut $\nabla(A)^+$ (or a separator A) in a network?

14. What is the value of a flow in a network no more than the capacity of any cut?

15. Explain the following proof.

2.1.2. LEMMA. *If f is any flow in D and C is any $s-t$ cut, then* $\mathrm{val}(f) \leq \mathrm{cap}(C)$.

PROOF. Let f and $C = \nabla^{+}(A)$ denote an arbitrary $s-t$ flow and an $s-t$ cut in D respectively. Then

$$\begin{aligned}
\mathrm{val}(f) &= \sum_{u} f(s,u) - \sum_{u} f(u,s) \\
&= \sum_{u} f(s,u) - \sum_{u} f(u,s) + \sum_{a \in A-s} \Big(\sum_{w} f(a,w) - \sum_{v} f(v,a)\Big) \\
&= \sum_{a \in A} \Big(\sum_{w} f(a,w) - \sum_{v} f(v,a)\Big) \\
&= \sum_{a \in A} \sum_{w} f(a,w) - \sum_{a \in A} \sum_{v} f(v,a) \\
&= \Big(\sum_{\substack{a \in A \\ w \in A}} f(a,w) + \sum_{\substack{a \in A \\ w \in V-A}} f(a,w)\Big) - \Big(\sum_{\substack{a \in A \\ v \in A}} f(v,a) + \sum_{\substack{a \in A \\ v \in V-A}} f(v,a)\Big)
\end{aligned}$$

Noting that the first and third terms cancel we have

$$\mathrm{val}(f) = \sum_{\substack{a \in A \\ w \in V-A}} f(a,w) - \sum_{\substack{a \in A \\ v \in V-A}} f(v,a).$$

But by definition of flow, $\sum_{a \in A, v \in V-A} f(v,a) \geq 0$, so

$$\mathrm{val}(f) \leq \sum_{\substack{a \in A \\ w \in V-A}} f(a,w) \leq \sum_{\substack{a \in A \\ w \in V-A}} c(a,w) \leq \mathrm{cap}(A).$$

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