LARSON—MATH 556—CLASSROOM WORKSHEET 21
The Birkhoff von Neumann Theorem

Review

- What is a *doubly stochastic matrix*?

- What is a bipartite graph that can be used to model which entries of a non-negative matrix have positive entries?

- What is a *permutation matrix*?

- What is a *convex combination* (of vectors, or matrices, etc)?

- What is the *Birkhoff von Neumann Theorem*?

Birkhoff's Algorithm

Let A_1 be a non-negative square matrix with constant (non-zero) row and column sums.

1. Let G_i be the associated bipartite graph (whose points represent the rows and columns of A_i and where ρ_j is adjacent to c_k if $(A_i)_{j,k}$ is non-zero).

2. Let M_i be a perfect matching in G_i.

3. Each line of M_i corresponds to an entry in A_i, each in a different row and different column. Let m_i be the minimum of these entries.

4. Let P_i be the permutation matrix with 1 entries in the coordinates corresponding to M_i.

5. Let $A_{i+1} = A_i - m_i P_i$.

6. If A_{i+1} is non-zero, repeat.

7. Else, if A_{i+1} is the zero matrix, then $A_1 = m_1 P_1 + m_2 P_2 + \ldots m_i P_i$.

Questions

1. Why does the correctness of Birkhoff's Algorithm prove the Birkhoff von Neumann Theorem?

2. What is a (vertex) *coloring* of a graph? What is a *valid* (or *proper*) coloring?

3. What is the *Four Color Theorem*?

4. What is the *maximum degree* Δ of a graph?

5. What is a *valid* (or *proper*) *line coloring* of a graph? What is χ_e?

6. What is Kőnig's Line Coloring Theorem?

7. What is an example of a Δ-coloring of a bipartite graph?

8. How can we *prove* Kőnig's Line Coloring Theorem?

9. What is a *regular* graph?

10. Why does a regular bipartite graph have a perfect matching?

11. Lovasz and Plummer claim that, given any bipartite graph G with maximum degree Δ there is a Δ-regular bipartite graph H where G is a subgraph of H. Why is that true?

12. Prove Kőnig's Line Coloring Theorem.