Last name ______________________

First name ______________________

LARSON—MATH 556—CLASSROOM WORKSHEET 20
The Birkhoff von Neumann Theorem

Review

- What is a *doubly stochastic matrix*?
- What is a bipartite graph that can be used to model which entries of a non-negative matrix have positive entries?
- What is a *permutation matrix*?
- What is a *convex combination* (of vectors, or matrices, etc)?
- What is the *Birkhoff von Neumann Theorem*?

Birkhoff's Algorithm

Let A_1 be a non-negative square matrix with constant (non-zero) row and column sums.

1. Let G_i be the associated bipartite graph (whose points represent the rows and columns of A_i and where ρ_j is adjacent to c_k if $(A_i)_{j,k}$ is non-zero).
2. Let M_i be a perfect matching in G_i.
3. Each line of M_i corresponds to an entry in A_i, each in a different row and different column. Let m_i be the minimum of these entries.
4. Let P_i be the permutation matrix with 1 entries in the coordinates corresponding to M_i.
5. Let $A_{i+1} = A_i - m_i P_i$.
6. If A_{i+1} is non-zero, repeat.
7. Else, if A_{i+1} is the zero matrix, then $A_1 = m_1 P_1 + m_2 P_2 + \ldots m_i P_i$.

Questions

1. Given a non-negative square matrix A with constant (non-zero) row and column sums, *why* does the associated graph G have a perfect matching M?

2. *Why* does each line of the matching M corresponds to an entry in A, each in a different row and different column?

3. *Why* is the minimum m of these entries non-zero?

4. *Why* is the matrix with 1 entries in the coordinates corresponding to M, and all other entries 0, a permutation matrix P?

5. Let $A' = A - mP$. Why is A' a matrix with constant row and column sums?

6. (**Claim:**) If A be a square nonnegative matrix with constant row and column sums then a finite number of iterations of Birkhoff's algorithm will yield a 0 matrix.

7. Why does the correctness of Birkhoff's Algorithm prove the Birkhoff von Neumann Theorem?

8. What is the *maximum degree* Δ of a graph?

9. What is a (proper) *line coloring* of a graph?

10. What is Kőnig's Line Coloring Theorem?

11. What is an example of a Δ-coloring of a bipartite graph?

12. How can we *prove* Kőnig's Line Coloring Theorem?