Last name ________________________

First name ________________________

LARSON—MATH 556—CLASSROOM WORKSHEET 17
Partially Ordered Sets & Dilworth's Theorem

Review

- A *partial order* on a set X is a relation "$\leq$" on X that is reflexive, anti-symmetric and transitive. We call $(X, \leq)$ a *partially ordered set* (or *poset*).

1. For $x, y, z \in X$, x *covers* y (or equivalently y *is covered* by x) if $y \leq x$ and $y \leq z \leq x$ implies that $z = x$ or $z = y$. A Hasse diagram (or covering diagram) for a poset $(X, \leq)$ is a representation of the elements of X together with a line between elements x and y is x covers y.
2. A *chain* in $(X, \leq)$ is a linearly ordered subset of X (with respect to the given order $\leq$), that is $C \subseteq X$ and $(C, \leq)$ is a linear order.
3. An *anti-chain* in a poset is a collection of elements which are pair-wise incomparable.
4. (**Divisibility poset**). Let $X = \{2, 3, \ldots 10\}$ and define the divisibility relation "$|$": For $x, y \in X$, $x|y$ (that is, x *divides* y, or y *is divisible by* x)) if there is an integer k such that $kx = y$. Show that $(X, |)$ is a poset.
5. We showed "$|$" is reflexive, anti-symmetric and transitive.

Dilworth's Theorem

1. Find examples of chains in $(X, |)$.

2. Find a partition of the elements of $(X, |)$ into a minimum number of chains.

3. Find a largest anti-chain in $(X, |)$.

Proof of Dilworth's Theorem

4. What is the bipartite graph G Lovasz and Plummer associate with a poset P?

5. Draw this graph for our divisibility poset example.

Let P be a poset with elements $X = \{x_1, \ldots, x_n\}$ and corresponding bipartite graph $G = (A, B)$.

6. (**Claim:**) Let M be a matching in G. Then there is a chain partition $\mathcal{P}$ of P such that $|M| + |\mathcal{P}| = n$.

7. (**Claim:**) If $C_G \subseteq A \cup B$ is a minimum cover of bigraph $G = (A, B)$ of poset P, then there is an antichain U contained in P with $|C_G| + |U| \geq n$.

8. (**Dilworth's Theorem:**) In any finite partially ordered set the cardinality of any largest antichain equals the cardinality of the smallest chain partition.