Last name ______________________

First name ______________________

LARSON—MATH 556—CLASSROOM WORKSHEET 15
Partially Ordered Sets & Dilworth's Theorem

Homework

- A *spanning tree* in a connected graph is a subgraph that is a tree that contains all the points of the parent graph. Prove that every connected graph has a spanning tree.

Review

- What is a *system of distinct representatives* (SDR) for a (finite) family of sets?
- What is *Hall's SDR Theorem*?

1. How can we apply the (bipartite graph version of) Hall's Theorem to model the SDR question and prove Hall's SDR Theorem?

Partially Ordered Sets & Dilworth's Theorem

A *partial order* on a set X is a relation "$\leq$" on X that is reflexive, anti-symmetric and transitive. We call $(X, \leq)$ a *partially ordered set* (or *poset*).

2. (**Inclusion poset**). Let $[4] = \{1, 2, 3, 4\}$ and $\mathcal{P}([4])$ be the *power set* of $[4]$. Consider this family of sets together with the inclusion relation "$\subseteq$": show that $(\mathcal{P}([4]), \subseteq)$ is a poset.

   (a) Show "$\subseteq$" is reflexive.

   (b) Show "$\subseteq$" is anti-symmetric.

   (c) Show "$\subseteq$" is transitive.

For $x, y, z \in X$, x *covers* y (or equivalently y *is covered* by x) if $y \leq x$ and $y \leq z \leq x$ implies that $z = x$ or $z = y$. A Hasse diagram (or covering diagram) for a poset $(X, \leq)$ is a representation of the elements of X together with a line between elements x and y is x covers y.

3. Draw the Hasse diagram for $(\mathcal{P}([4]), \subseteq)$.

A *chain* in $(X, \leq)$ is a linearly ordered subset of X (with respect to the given order $\leq$), that is $C \subseteq X$ and $(C, \leq)$ is a linear order.

4. Find examples of chains in $(\mathcal{P}([4]), \subseteq)$.

5. Find a partition of the elements of $(\mathcal{P}([4]), \subseteq)$ into a minimum number of chains.

An *anti-chain* in $(\mathcal{P}([4]), \subseteq)$ is a collection of elements which are pair-wise incomparable.

6. Find a largest anti-chain in $(\mathcal{P}([4]), \subseteq)$.

7. (**Divisibility poset**). Let $X = \{2, 3, \ldots 10\}$ and define the divisibility relation "$|$": For $x, y \in X$, $x|y$ (that is, *x divides y*, or *y is divisible by x*)) if there is an integer k such that $kx = y$. Show that $(X, |)$ is a poset.

   (a) Show "$|$" is reflexive.

   (b) Show "$|$" is anti-symmetric.

(c) Show “$|$” is transitive.

8. Draw the Hasse diagram for $(X, |)$.

9. Find examples of chains in $(X, |)$.

10. Find a partition of the elements of $(X, |)$ into a minimum number of chains.

11. Find a largest anti-chain in $(X, |)$.