Last name ______________________

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LARSON—MATH 556—CLASSROOM WORKSHEET 14
Hungarian Forests and the Hungarian Method

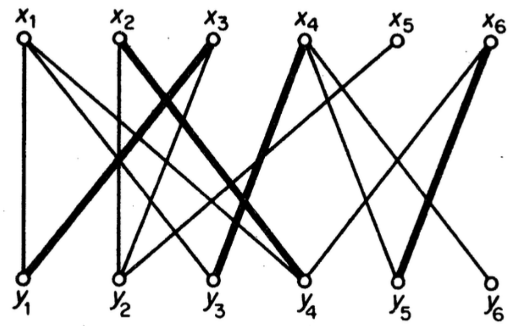
Concepts & Notation

- *assignment problem*, *graph* G, *points* $V(G)$, *lines* $E(G)$, *adjacent*, *incident*.
- *line covering*, *line covering number* ρ, *matching*, *matching number* ν, point covering, *point covering number* τ, *independent set*, *independence number* α.

Review

- What is **König's Minimax Theorem**?
- What is **Berge's Theorem**?
- **Hungarian Forest**. For a bipartite graph $G = (A, B)$, and matching M, and M-unsaturated vertices $A_1 \subseteq A$ and $B_1 \subseteq B$, a *Hungarian forest* F is a forest subgraph of G that is maximal with respect to the following conditions:
  - Every B-point in F is either in B_1 or has degree 2 with one incident edge in M,
  - Each tree component of F contains a single point in A_1.

- **Hungarian Method** For a bipartite graph $G = (A, B)$, and matching M, the *Hungarian Method* produces, on each iteration, either a larger matching or a point cover with $|M|$ points.
  1. Find a (maximal) Hungarian forest F with respect to M.
  2. If F contains a point w in B_1 then there is an M-augmenting path P, the unique path in the tree component containing w from w to the unique point v in A_1 in that component. Let $M' = P \oplus M$ be the updated (larger) matching, and repeat.
  3. If F contains no points in B_1 then let $X = A \setminus V(F)$ and $Y = B \cap V(F)$. $X \cup Y$ is a minimum point cover (with cardinality $|M|$).
- Why does a (maximal) Hungarian forest F contain $|A_1|$ tree components?
- Why is every path from a point in A_1 to a leaf (pendant, degree-1 point) in a Hungarian forest F an M-alternating path?
- If G has $2n$ points, why does the Hungarian Method terminate in at most n iterations?
- If a produced Hungarian forest F contains a point in B_1, why must there be an M-augmenting path in G?
- If F is a (maximal) Hungarian forest with no point in B_1, why is $X \cup Y$ a point cover?



1. If F is a (maximal) Hungarian forest with no point in B_1, why does $|X \cup Y| = |M|$?

2. Why does the Hungarian method produce a *provably* maximum matching in a bipartite graph?

Hall's SDR Theorem

3. What is a *system of distinct representatives* (SDR) for a (finite) family of sets?

4. What is *Hall's SDR Theorem*?

5. How can we apply the (bipartite graph version of) Hall's Theorem to model the SDR question and prove Hall's SDR Theorem?

Partially Ordered Sets & Dilworth's Theorem

A *partial order* on a set X is a relation "$\leq$" on X that is reflexive, anti-symmetric and transitive. We call $(X, \leq)$ a *partially ordered set* (or *poset*).

6. Let $X = \{2, 3, \ldots 10\}$ and define the divisibility relation "|": For $x, y \in X$, $x|y$ (that is, x *divides* y, or y *is divisible by* x)) if there is an integer k such that $kx = y$.

   Show that $(X, |)$ is a poset.

   (a) Show "|" is reflexive.

   (b) Show "|" is anti-symmetric.

   (c) Show "|" is transitive.

   For $x, y, z \in X$, x *covers* y (or equivalently y *is covered* by x) if $y \leq x$ and $y \leq z \leq x$ implies that $z = x$ or $z = y$.

   A Hasse diagram (or covering diagram) for a poset $(X, \leq)$ is a representation of the elements of X together with a line between elements x and y is x covers y.

7. Draw the Hasse diagram for $(X, |)$.