Last name ______________________

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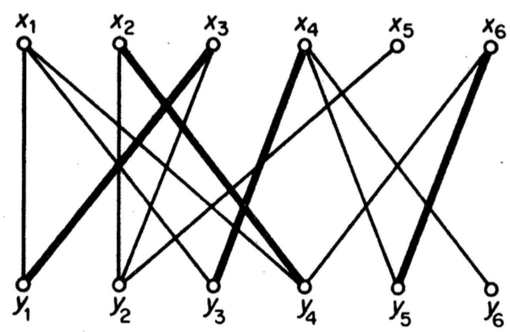
LARSON—MATH 556—CLASSROOM WORKSHEET 13
Hungarian Forests and the Hungarian Method

Concepts & Notation

- *assignment problem*, *graph* G, *points* $V(G)$, *lines* $E(G)$, *adjacent*, *incident*.
- *line covering*, *line covering number* ρ, *matching*, *matching number* ν, point covering, *point covering number* τ, *independent set*, *independence number* α.

Review

- **König's Theorem:** For any bipartite graph, $\tau = \nu$.
- Let M be a matching in a graph. What is an M-*alternating path*?
- Let M be a matching in a graph. What is an M-*augmenting path*?
- What is Berge's Theorem?



Hungarian Forest. For a bipartite graph $G = (A, B)$, and matching M, and M-unsaturated vertices $A_1 \subseteq A$ and $B_1 \subseteq B$, a *Hungarian forest* F is a forest subgraph of G that is maximal with respect to the following conditions:

- Every B-point in F is either in B_1 or has degree 2 with one incident edge in M,
- Each tree component of F contains a single point in A_1.

1. Why does a (maximal) Hungarian forest F contain $|A_1|$ tree components?

2. Why is every path in a Hungarian forest F an M-alternating path?

Hungarian Method For a bipartite graph $G = (A, B)$, and matching M, the *Hungarian Method* produces, on each iteration, either a larger matching or a point cover with $|M|$ points.

(a) Find a (maximal) Hungarian forest F with respect to M.

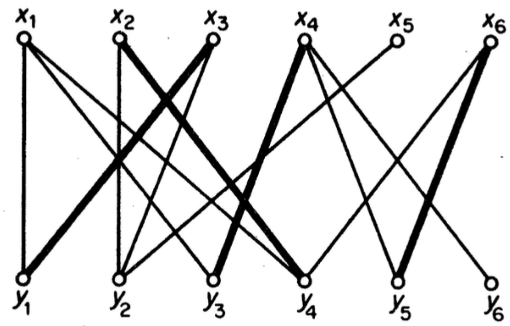
(b) If F contains a point w in B_1 then there is an M-augmenting path P, the unique path in the tree component containing w from w to the unique point v in A_1 in that component. Let $M' = P \oplus M$ be the updated (larger) matching, and repeat.

(c) If F contains no points in B_1 then let $X = A \setminus V(F)$ and $Y = B \cup V(F)$. $X \cup Y$ is a minimum point cover (with cardinality $|M|$).

3. If G has $2n$ points, why does the Hungarian Method terminate in at most n iterations?

4. If a produced Hungarian forest F contains a point in B_1, why must there be an M-augmenting path in G?

5. If F is a (maximal) Hungarian forest with no point in B_1, why is $X \cup Y$ a point cover?



6. If F is a (maximal) Hungarian forest with no point in B_1, why does $|X \cup Y| = |M|$?

7. Why does the Hungarian method produce a *provably* maximum matching in a bipartite graph?

Hall's SDR Theorem

8. What is a *system of distinct representatives* (SDR) for a (finite) family of sets?

9. What is *Hall's SDR Theorem*?

10. How can we apply the (bipartite graph version of) Hall's Theorem to model the SDR question and prove Hall's SDR Theorem?