Last name ______________________

First name ______________________

LARSON—MATH 556—CLASSROOM WORKSHEET 01
The Assignment Problem & Matchings.

The Assignment Problem. Let $\{u_1,\dots,u_n\}$ be the set of employees and $\{v_1,\dots,v_n\}$ be the set of jobs that must get done. Each employee u_i able to do some subset $N(u_i)$ of the jobs. Is there a possible *assignment* of the employees to the jobs so that all the jobs get done?

1. (Ex. 1) There are 4 employees: u_1, u_2, u_3, and u_4, and 4 jobs: v_1, v_2, v_3 and v_4. u_1 can do jobs v_1, v_2 and v_3. u_2 can do jobs v_2, v_3 and v_4. u_3 can do jobs v_3, v_4 and v_1. u_4 can do jobs v_4, v_1 and v_2. Is there a possible assignment of the employees to the jobs so that all the jobs get done?

2. (Ex. 2) u_1 can do jobs v_2 and v_3. u_2 can do jobs v_1, v_2, v_3 and v_4. u_3 can do jobs v_2 and v_3. u_4 can do jobs v_2 and v_3. Is there a possible assignment of the employees to the jobs so that all the jobs get done?

A **graph** G consists of a set of **points** $V(G)$ and **lines** $E(G)$ which are pairs $\{v,w\}\subseteq V(G)$ or, simply, vw. If vw is a line, point v is said to be **adjacent** to point w, while v is **incident** to line vw.

We can represent an assignment problem with a graph G where $V(G)=\{u_1,\dots,u_n,v_1,\dots,v_n\}$ and $E(G)=\{u_iv_j : u_i \text{ can do job } v_j\}$.

3. Represent the situation in Ex. 1 as a graph.

4. Represent the situation in Ex. 2 as a graph.

We can also represent an assignment problem with a matrix A with n rows representing the employees and n columns representing the jobs where $A_{ij} = 1$ if employee u_i can do job v_j. Otherwise let $A_{ij} = 0$.

5. Represent the situation in Ex. 1 as a matrix.

6. Represent the situation in Ex. 2 as a matrix.

7. When a solution for an assignment problem exists you can *prove* it by specifying a *feasible* assignment of employees to jobs. Prove there is a solution for Ex. 1.

8. When a solution for an assignment problem does **not** exist you can *prove* that by specifying a set of employees S where, altogether, they are able to do fewer than $|S|$ jobs. Prove there is no solution for Ex. 2.

A **bipartite graph** is a graph G where the points $V(G)$ can be partitioned into two sets A and B such that every edge has one endpoint in A and the other in B. In the assignment problem, the corresponding graph is bipartite: let A be the set of employees and B be the set of jobs.

9. To define a graph on a set of points, it is enough to define an *adjacency relation* which specifies which vertices are adjacent. $V(G) = \{1, 2, 3, 4, 5\}$. Let a be adjacent to b in G if and only if $a + b$ is even. Is G bipartite?

10. Let A be any set and B be the set of subsets of A. Let $V(G) = A \cup B$, and let $a \in A$ be adjacent to $b \in B$ if and only if $a \in b$. Let $A = \{a_1, a_2\}$. Find B and then draw G. Is G bipartite?

11. Let A be any set and B be the set of subsets of A. Let $V(G) = A \cup B$, and let $a \in A$ be adjacent to $b \in B$ if and only if $a \in b$. Let $A = \{a_1, a_2, a_3\}$. Find B and then draw G. Is G bipartite?

Matchings. A **matching** in a graph is a set of lines where no pair of lines is incident to the same point. A matching is *maximum* if there is no matching with larger cardinality. The cardinality of a maximum matching in graph G is the **matching number** $\nu(G)$.

12. Find a maximum matching for the graph in Ex. 1. Then find ν. Can you find a matching which is maxim*al* (can't be extended) but is not maxim*um*?

13. Find a maximum matching for the graph in Ex. 2. Then find ν. Can you find a matching which is maxim*al* (can't be extended) but is not maxim*um*?