Last name ______________________

First name ______________________

LARSON—MATH 356—CLASSROOM WORKSHEET 07
Orders of Magnitude

Review

- Explain why 2^x grows *faster* than x^α for any positive α.
- Explain why x^α grows *faster* than $\log x$ for any positive α.

  Definition (little-o). We say that $f(x) = o(g(x))$ $(x \to \infty)$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ exists and is equal to 0.

Orders of Magnitude

Definition (big-O). We say that $f(x) = O(g(x))$ $(x \to \infty)$ if $\exists C, x_0$ such that $|f(x)| < Cg(x)$ $(\forall x > x_0)$.

1. Check these examples:

   (a) $x^3 + 5x^2 + 77\cos x = O(x^5)$.

   (b) $\frac{1}{(1+x^2)} = O(1)$.

   Definition (Theta). We say that $f(x) = \Theta(g(x))$ if there are constants $c1 > 0$, $c2 > 0$, x_0 such that for all $x > x_0$, it is true that $c_1 g(x) < f(x) < c_2 g(x)$.

2. Check these examples:

   (a) $(x+1)^2 = \Theta(3x^2)$.

   (b) $\frac{(x^2+5x+7)}{(5x^3+7x+2)} = \Theta(\frac{1}{x})$.

   (c) $\sqrt{3+\sqrt{2x}} = \Theta(x^{\frac{1}{4}})$.

   (d) $(1+\frac{3}{x})^x = \Theta(1)$.

Definition (twiddles). We say that $f(x) \sim g(x)$ if $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = 1$.

(a) $x^2 + x \sim x^2$.

(b) $(3x+1)^4 \sim 81x^4$.

(c) $\sin\frac{1}{x} \sim \frac{1}{x}$.

(d) $\frac{(2x^3+5x+7)}{(x^2+4)} \sim 2x$.

(e) $2^x + 7\log x + \cos x \sim 2^x$.

The last symbol in the asymptotic set that we will need is the 'Ω.' In a nutshell, 'Ω' is the negation of 'o.' That is to say, $f(x) = \Omega(g(x))$ means that it is not true that $f(x) = o(g(x))$. In the study of algorithms for computers, the 'Ω' is used when we want to express the thought that a certain calculation takes at least so-and-so long to do.

Definition (Omega). We say that $f(x) = \Omega(g(x))$ if there is an $\epsilon > 0$ and a sequence $x_1, x_2, x_3, \ldots \to \infty$ such that $\forall j : |f(x_j)| > \epsilon g(x_j)$.

3. Show $2^x = \Omega(x^\alpha)$ (for any $\alpha > 0$).

Definition. A function f is of **exponential growth** if there exists $c > 1$ such that $f(x) = \Omega(c^x)$ and there exists d such that $f(x) = O(d^x)$.

4. Show that $f(x) = 2^x$ grows exponentially according to this definition.

5. How about $f(x) = 2^x + x^{100}$?