

Right inverses

- ▶ a matrix X that satisfies $AX = I$ is a *right inverse* of A
- ▶ if a right inverse exists we say that A is *right-invertible*
- ▶ A is right-invertible if and only if A^T is left-invertible:

$$AX = I \iff (AX)^T = I \iff X^T A^T = I$$

- ▶ so we conclude
 A is right-invertible if and only if its rows are linearly independent
- ▶ right-invertible matrices are wide or square

Solving linear equations with a right inverse

- ▶ suppose A has a right inverse B
- ▶ consider the (square or underdetermined) equations $Ax = b$
- ▶ $x = Bb$ is a solution:

$$Ax = A(Bb) = (AB)b = Ib = b$$

- ▶ so $Ax = b$ has a solution for *any* b

Example

- ▶ same A , B , C in example above
- ▶ C^T and B^T are both right inverses of A^T
- ▶ under-determined equations $A^T x = (1, 2)$ has (different) solutions

$$B^T(1, 2) = (1/3, 2/3, -2/3), \quad C^T(1, 2) = (0, 1/2, -1)$$

(there are many other solutions as well)

Outline

Left and right inverses

Inverse

Solving linear equations

Examples

Pseudo-inverse

Inverse

- ▶ if A has a left and a right inverse, they are unique and equal (and we say that A is *invertible*)
- ▶ so A must be square
- ▶ to see this: if $AX = I$, $YA = I$

$$X = IX = (YA)X = Y(AX) = YI = Y$$

- ▶ we denote them by A^{-1} :

$$A^{-1}A = AA^{-1} = I$$

- ▶ inverse of inverse: $(A^{-1})^{-1} = A$

Solving square systems of linear equations

- ▶ suppose A is invertible
- ▶ for any b , $Ax = b$ has the unique solution

$$x = A^{-1}b$$

- ▶ matrix generalization of simple scalar equation $ax = b$ having solution $x = (1/a)b$ (for $a \neq 0$)
- ▶ simple-looking formula $x = A^{-1}b$ is basis for many applications

Invertible matrices

the following are equivalent for a square matrix A :

- ▶ A is invertible
- ▶ columns of A are linearly independent
- ▶ rows of A are linearly independent
- ▶ A has a left inverse
- ▶ A has a right inverse

if any of these hold, all others do

Examples

- ▶ $I^{-1} = I$
- ▶ if Q is orthogonal, *i.e.*, square with $Q^T Q = I$, then $Q^{-1} = Q^T$
- ▶ 2×2 matrix A is invertible if and only $A_{11}A_{22} \neq A_{12}A_{21}$

$$A^{-1} = \frac{1}{A_{11}A_{22} - A_{12}A_{21}} \begin{bmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{bmatrix}$$

- you need to know this formula
- there are similar but *much* more complicated formulas for larger matrices (and no, you do not need to know them)

Non-obvious example

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 2 & 2 \\ -3 & -4 & -4 \end{bmatrix}$$

- ▶ A is invertible, with inverse

$$A^{-1} = \frac{1}{30} \begin{bmatrix} 0 & -20 & -10 \\ -6 & 5 & -2 \\ 6 & 10 & 2 \end{bmatrix}.$$

- ▶ verified by checking $AA^{-1} = I$ (or $A^{-1}A = I$)
- ▶ we'll soon see how to compute the inverse

Properties

- ▶ $(AB)^{-1} = B^{-1}A^{-1}$ (provided inverses exist)
- ▶ $(A^T)^{-1} = (A^{-1})^T$ (sometimes denoted A^{-T})
- ▶ negative matrix powers: $(A^{-1})^k$ is denoted A^{-k}
- ▶ with $A^0 = I$, identity $A^k A^l = A^{k+l}$ holds for any integers k, l

Triangular matrices

- ▶ lower triangular L with nonzero diagonal entries is invertible
- ▶ so see this, write $Lx = 0$ as

$$\begin{array}{rcl} L_{11}x_1 & = & 0 \\ L_{21}x_1 + L_{22}x_2 & = & 0 \\ & \vdots & \\ L_{n1}x_1 + L_{n2}x_2 + \cdots + L_{n,n-1}x_{n-1} + L_{nn}x_n & = & 0 \end{array}$$

- from first equation, $x_1 = 0$ (since $L_{11} \neq 0$)
- second equation reduces to $L_{22}x_2 = 0$, so $x_2 = 0$ (since $L_{22} \neq 0$)
- and so on

this shows columns of L are linearly independent, so L is invertible

- ▶ upper triangular R with nonzero diagonal entries is invertible

Inverse via QR factorization

- ▶ suppose A is square and invertible
- ▶ so its columns are linearly independent
- ▶ so Gram–Schmidt gives QR factorization
 - $A = QR$
 - Q is orthogonal: $Q^T Q = I$
 - R is upper triangular with positive diagonal entries, hence invertible
- ▶ so we have

$$A^{-1} = (QR)^{-1} = R^{-1}Q^{-1} = R^{-1}Q^T$$