

Matrix powers

- ▶ for A square, A^2 means AA , and same for higher powers
- ▶ with convention $A^0 = I$ we have $A^k A^l = A^{k+l}$
- ▶ negative powers later; fractional powers in other courses

Outline

Matrix multiplication

Composition of linear functions

Matrix powers

QR factorization

Gram–Schmidt in matrix notation

- ▶ run Gram–Schmidt on columns $a_1, \dots, a_k$ of $n \times k$ matrix A
- ▶ if columns are linearly independent, get orthonormal $q_1, \dots, q_k$
- ▶ define $n \times k$ matrix Q with columns $q_1, \dots, q_k$
- ▶ $Q^T Q = I$
- ▶ from Gram–Schmidt algorithm

$$\begin{aligned} a_i &= (q_1^T a_i)q_1 + \dots + (q_{i-1}^T a_i)q_{i-1} + \|\tilde{q}_i\|q_i \\ &= R_{1i}q_1 + \dots + R_{ii}q_i \end{aligned}$$

with $R_{ij} = q_i^T a_j$ for $i < j$ and $R_{ii} = \|\tilde{q}_i\|$

- ▶ defining $R_{ij} = 0$ for $i > j$ we have $A = QR$
- ▶ R is upper triangular, with positive diagonal entries

QR factorization

- ▶ $A = QR$ is called *QR factorization* of A
- ▶ factors satisfy $Q^T Q = I$, R upper triangular with positive diagonal entries
- ▶ can be computed using Gram–Schmidt algorithm (or some variations)
- ▶ has a *huge* number of uses, which we'll see soon

11. Matrix inverses

Outline

Left and right inverses

Inverse

Solving linear equations

Examples

Pseudo-inverse

Left inverses

- ▶ a number x that satisfies $xa = 1$ is called the inverse of a
- ▶ inverse (*i.e.*, $1/a$) exists if and only if $a \neq 0$, and is unique
- ▶ a matrix X that satisfies $XA = I$ is called a *left inverse* of A
- ▶ if a left inverse exists we say that A is *left-invertible*
- ▶ example: the matrix

$$A = \begin{bmatrix} -3 & -4 \\ 4 & 6 \\ 1 & 1 \end{bmatrix}$$

has two different left inverses:

$$B = \frac{1}{9} \begin{bmatrix} -11 & -10 & 16 \\ 7 & 8 & -11 \end{bmatrix}, \quad C = \frac{1}{2} \begin{bmatrix} 0 & -1 & 6 \\ 0 & 1 & -4 \end{bmatrix}$$

Left inverse and column independence

- ▶ if A has a left inverse C then the columns of A are linearly independent
- ▶ to see this: if $Ax = 0$ and $CA = I$ then

$$0 = C0 = C(Ax) = (CA)x = Ix = x$$

- ▶ we'll see later the converse is also true, so
a matrix is left-invertible if and only if its columns are linearly independent
- ▶ matrix generalization of
a number is invertible if and only if it is nonzero
- ▶ so left-invertible matrices are tall or square

Solving linear equations with a left inverse

- ▶ suppose $Ax = b$, and A has a left inverse C
- ▶ then $Cb = C(Ax) = (CA)x = Ix = x$
- ▶ so multiplying the right-hand side by a left inverse yields the solution

Example

$$A = \begin{bmatrix} -3 & -4 \\ 4 & 6 \\ 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

- ▶ over-determined equations $Ax = b$ have (unique) solution $x = (1, -1)$
- ▶ A has two different left inverses,

$$B = \frac{1}{9} \begin{bmatrix} -11 & -10 & 16 \\ 7 & 8 & -11 \end{bmatrix}, \quad C = \frac{1}{2} \begin{bmatrix} 0 & -1 & 6 \\ 0 & 1 & -4 \end{bmatrix}$$

- ▶ multiplying the right-hand side with the left inverse B we get

$$Bb = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

- ▶ and also

$$Cb = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Right inverses

- ▶ a matrix X that satisfies $AX = I$ is a *right inverse* of A
- ▶ if a right inverse exists we say that A is *right-invertible*
- ▶ A is right-invertible if and only if A^T is left-invertible:

$$AX = I \iff (AX)^T = I \iff X^T A^T = I$$

- ▶ so we conclude
 A is right-invertible if and only if its rows are linearly independent
- ▶ right-invertible matrices are wide or square

Solving linear equations with a right inverse

- ▶ suppose A has a right inverse B
- ▶ consider the (square or underdetermined) equations $Ax = b$
- ▶ $x = Bb$ is a solution:

$$Ax = A(Bb) = (AB)b = Ib = b$$

- ▶ so $Ax = b$ has a solution for *any* b

Example

- ▶ same A , B , C in example above
- ▶ C^T and B^T are both right inverses of A^T
- ▶ under-determined equations $A^T x = (1, 2)$ has (different) solutions

$$B^T(1, 2) = (1/3, 2/3, -2/3), \quad C^T(1, 2) = (0, 1/2, -1)$$

(there are many other solutions as well)