LARSON—MATH 310—Homework 14
Test 2 REVIEW

On all questions you will be asked to **show your work**. You may be asked to **explain** your answers.

Definitions, Theorems, Results, Ideas. State the following.

- State both parts of the Fundamental Theorem of Linear Algebra.
- State the formula for the projection $\vec{p}$ of vector $\vec{b}$ on the subspace of vectors $\vec{a_1}, \ldots, \vec{a_n}$
- State the main idea of the Gram-Schmidt algorithm.
- State the definition of the determinant of a square triangular matrix.
- State the two rules for calculating the determinant of an arbitrary square matrix.
- State definitions for symmetric, orthogonal, and positive semi-definite matrices. Give an example of each kind.

Problems.

1. Let vector $\vec{b} = (3, 4, 4)$. Find the projection $\vec{p}$ onto the line through vector $\vec{a} = (2, 2, 1)$.
2. Find the projection $\vec{p}$ onto the subspace containing vectors $\vec{a_1} = (2, 2, 1)$ and $\vec{a_2} = (1, 0, 0)$.
3. Find an orthogonal basis for the subspace formed by vectors $\vec{b} = (3, 4, 4)$ and $\vec{a} = (2, 2, 1)$.
4. Find $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 0 & 0 & 5 \end{vmatrix}, \begin{vmatrix} 1 & 1 & 1 \\ 0 & 0 & 5 \\ 2 & 2 & 2 \end{vmatrix}, \begin{vmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 0 & 0 & 7 \end{vmatrix}, \begin{vmatrix} 4 & 5 & 6 & 7 \\ 1 & 2 & 3 & 4 \\ 8 & 9 & 9 & 8 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 4 & 0 \end{vmatrix}.$

   Let $A = \begin{bmatrix} 0 & 2 \\ 2 & 3 \end{bmatrix}$.

5. Find the eigenvalues and corresponding eigenvectors of A.

   Let $A = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$.

6. A is an $m \times n$ matrix. Find m and n. The following algorithm works for any $m \times n$ matrix!
7. Find the rank r of A. (This tells you how many vectors are in the row space and null space of A). We proved that this is also the rank of $A^T A$ and AA^T.

$A = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$ (repeated from last page).

Given any $m \times n$ matrix A we can find orthogonal matrices U and V and a diagonal matrix Σ so that $A = U\Sigma V^T$. This is it, the Singular Value Decomposition (SVD).

Furthermore $V = [\vec{v_1} \dots \vec{v_r} \dots \vec{v_n}]$ (where r is the rank of A), Σ is an $m \times n$ 0s matrix with positive numbers $\sigma_1, \dots, \sigma_r$ on the diagonal, $A\vec{v_i} = \sigma_i \vec{u_i}$, $\sigma_i = \|A\vec{v_i}\|$, and $U = [\vec{u_1} \dots \vec{u_r} \dots \vec{u_m}]$, where $\vec{u_i} = \frac{1}{\sigma_i} A\vec{v_i}$ for $i = 1, \dots r$.

8. Find $A^T A$.
9. Find the eigenvalues of $A^T A$. There will be r positive eigenvalues: $\sigma_1^2, \dots, \sigma_r^2$.
10. Find the corresponding eigenvectors for these eigenvalues (of $A^T A$) and normalize them. Call these: $\vec{v_1}, \dots, \vec{v_r}$. (We proved in class that they must be orthogonal).
11. Normalize the vectors corresponding to the 0-eigenvalues of $A^T A$. Call these $v_{r+1}, \dots, v_n$. (In the general case you need to use Gram-Schmidt to find an orthonormal basis of these.)
12. Let $V = \begin{bmatrix} v_1 \dots v_r \dots v_n \end{bmatrix}$.
13. Show that V is orthogonal.
14. For each $i \in \{1, \dots, r\}$, find $\|A\vec{v_i}\|$. Check that $\sigma_i = \|A\vec{v_i}\|$.
15. Let Σ be the $m \times n$ matrix with the σ_i's on the diagonal for $i = 1, \dots, r$, and 0 for every other entry.
16. Find AA^T.
17. Find the eigenvalues of AA^T. There will be r positive eigenvalues and they should be the same as the ones for $A^T A$: $\sigma_1^2, \dots, \sigma_r^2$.
18. Find the corresponding eigenvectors for these eigenvalues (of AA^T) and normalize them. Call these: $\vec{u_1}, \dots, \vec{u_r}$. (We proved in class that they must be orthogonal).
19. Let $U = \begin{bmatrix} u_1 \dots u_r \dots u_m \end{bmatrix}$.
20. Show that U is orthogonal.
21. For each $i = 1, \dots, r$, check that $A\vec{v_i} = \sigma_i \vec{u_i}$.
22. Show that $A = U\Sigma V^T$.