LARSON—MATH 310—Homework 13
SVD!

Show all your work.

Let $A = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$.

1. A is an $m \times n$ matrix. Find m and n and the rank r of A.
2. Find $A^T A$.
3. Check that the rank of $A^T A$ is the same as the rank r of A.
4. Find the eigenvalues of $A^T A$. There will be r positive eigenvalues: $\sigma_1^2, \ldots, \sigma_r^2$. (Arrange these so $\sigma_1^2 \geq \sigma_2^2 \geq \ldots \geq \sigma_r^2$).
5. Find the *singular* values $\sigma_1, \ldots, \sigma_r$.
6. Find the corresponding eigenvectors for the r positive eigenvalues of $A^T A$ and normalize them. Call these: $\vec{v_1}, \ldots, \vec{v_r}$.
7. Check that your eigenvalue-eigenvector pairs work.
8. The eigenvectors corresponding to the 0-eigenvalues of $A^T A$ are a basis for the null space of $A^T A$. They might not be orthogonal. Use Gram-Schmidt to convert these to an orthogonal (actually, orthonormal) basis for the null space of $A^T A$. Call these $v_{r+1}, \ldots, v_n$.
9. What is an *orthogonal* matrix?
10. Let $V = \begin{bmatrix} v_1 \ldots v_r \ldots v_n \end{bmatrix}$, check that V is $n \times n$, and show that V is orthogonal.
11. For each $i \in \{1, \ldots, r\}$, find $\|A\vec{v_i}\|$.
12. Let Σ be the $m \times n$ matrix with the σ_i's on the diagonal for $i = 1, \ldots, r$, and 0 for every other entry. Find Σ.
13. Find AA^T.
14. Check that the rank of AA^T is the same as the rank r of A.
15. Find the eigenvalues of AA^T. Call the positive ones: $\sigma_1^2, \ldots, \sigma_r^2$.
16. Find the corresponding eigenvectors for the eigenvalues of AA^T and normalize them. Call the eigenvectors corresponding to the r positive eigenvalues: $\vec{u_1}, \ldots, \vec{u_r}$.
17. Check that your eigenvalue-eigenvector pairs work.
18. The eigenvectors corresponding to the 0-eigenvalues of AA^T are the null space of AA^T. We want an orthogonal basis. Find any basis and use Gram-Schmidt to convert these to an orthogonal (actually, orthonormal) basis for the null space of AA^T. Call these $v_{r+1}, \ldots, v_n$.
19. Let $U = \begin{bmatrix} u_1 \ldots u_r \ldots u_m \end{bmatrix}$, check that U is $m \times m$, and show that U is orthogonal.
20. For each $i = 1, \ldots, r$, show that $A\vec{v_i} = \sigma_i \vec{u_i}$.