Last name ______________________

First name ______________________

LARSON—MATH 310—CLASSROOM WORKSHEET 28
SVD & Rank-1 Matrices

Let $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

The SVD here is $A = U\Sigma V^T$:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

1. What is m, n and the rank r of the given matrix A?

2. Do the sizes of U, Σ and V make sense?

3. How many positive singular values must there be?

4. Call the positive singular values $\sigma_1, \ldots, \sigma_r$. What are the positive singular values $\sigma_1 \geq \ldots \geq \sigma_r$? Find Σ.

5. What are the $\vec{v_i}$'s?

6. Find V

7. Check that V is orthogonal.

8. What are the $\vec{u_i}$'s?

9. Check that U is orthogonal.

10. Check that $A\vec{v_i} = \sigma_i \vec{u_i}$ for $i = 1, \ldots, r$. Check that $A\vec{v_i} = 0$ for $i > r$.

11. Find the matrices $\sigma_1 \vec{u_1} \vec{v_1}^T$, $\ldots$, $\sigma_r \vec{u_r} \vec{v_r}^T$.

12. Check that the rank of each of $\sigma_1 \vec{u_1} \vec{v_1}^T$, $\ldots$, $\sigma_r \vec{u_r} \vec{v_r}^T$ is 1.

13. Check that $A = \sigma_1 \vec{u_1} \vec{v_1}^T + \ldots + \sigma_r \vec{u_r} \vec{v_r}^T$ (that is, that A is a sum of r rank-1 matrices).