Last name ______________________

First name ______________________

LARSON—MATH 310—CLASSROOM WORKSHEET 26
Singular Value Decomposition.

Review

- If there is a non-zero vector $\vec{x}$ and scalar λ with $A\vec{x} = \lambda\vec{x}$ then λ is an **eigenvalue** of A and $\vec{x}$ is a corresponding **eigenvector**.
- If $A\vec{x} = \lambda\vec{x}$, then $A\vec{x} - \lambda\vec{x} = 0$, and $(A - \lambda I)\vec{x} = 0$. Since $\vec{x}$ is non-zero that means that $(A - \lambda I)$ is not invertible, that the RREF has a 0-row, and that $\det(A) = 0$.
- (**Claim**:) The eigenvalues of any symmetric matrix are real.
- (**Claim**:) Any symmetric matrix A can be written as $A = Q\Lambda Q^T$, where Λ is diagonal and Q is orthogonal. (This is the **Real Spectral Theorem**).
- (**Claim**:) If A is a symmetric matrix then eigenvectors corresponding to different eigenvalues are orthogonal.
- (**Claim**:) For any matrix A, the eigenvalues of A^TA and AA^T are non-negative.

Let $A = \begin{bmatrix} 3 & 0 \\ 4 & 0 \end{bmatrix}$.

We will find the **singular value decomposition** (SVD) of an $m \times n$ matrix A. That is we want to write $A = U\Sigma V^T$ where U and V are orthogonal matrices and Σ is a "diagonal" matrix (all-zeros except singular values on the diagonal).

1. What is m, n and the rank r of the given matrix A?

2. If A is 2×2, what sizes do U, Σ and V have to be?

3. Why must $AV = U\Sigma$?

4. If $A[\vec{v_1}\vec{v_2}] = [\vec{u_1}\vec{u_2}] \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}$, what can we say about $A\vec{v_i}$? (The σ_i's are the singular values).

5. Find AA^T and check that its rank is r.

6. We proved AA^T is symmetric with nonnegative eigenvalues (this means it is *positive semi-definite*). Call the r positive eigenvalues: $\sigma_1^2, \dots, \sigma_r^2$, with $\sigma_1^2 \geq \dots \geq \sigma_r^2$ (The rest must be 0).

7. What are the singular values? Find Σ.

8. If there are any zero eigenvalues of AA^T, there will be $n-r$ of them, and we will find a basis for the nullspace, then use Gram-Schmidt to get an orthonormal basis for the null space and call those vectors $v_{\vec{r+1}}, \dots, \vec{n}$. What is the situation here?

9. Find unit (normalized) eigenvectors $\vec{v_1}, \dots, \vec{v_r}$ corresponding to the eigenvalues of AA^T. Let $V = [\vec{v_1} \dots \vec{v_r} v_{\vec{r+1}} \dots \vec{v_n}]$.

10. Check that V is orthogonal.

11. Find A^TA and check that its rank is r. We proved A^TA is symmetric with nonnegative eigenvalues (this means it is also *positive semi-definite*). Why will it have the same eigenvalues as AA^T?