Last name ______________________

First name ______________________

LARSON—MATH 310—CLASSROOM WORKSHEET 20
Orthogonal Bases & Gram-Schmidt.

Review

We found that, for any matrix A (with linearly independent columns), that the projection $\vec{p}$ of a vector $\vec{b}$ onto the column space of A is:

$$\vec{p} = A\vec{x} \text{ where } \vec{x} = (A^TA)^{-1}A^T\vec{b}.$$

$$\text{so } \vec{p} = A(A^TA)^{-1}A^T\vec{b}.$$

We can use this idea repeatedly to convert and collection of linearly independent vectors $\vec{a_1}, \vec{a_2}, \ldots, \vec{a_k}$ (which are a basis for the space those vectors span to a *nice* basis for the same space: orthonormal vectors $\vec{a_1}', \vec{a_2}', \ldots, \vec{a_k}'$).

Here's the idea informally:

1. Unit-ize $\vec{a_1}$ to get $\vec{a_1}'$.
2. Project vector $\vec{a_2}$ onto the matrix A with $\vec{a_1}'$ as its only column.
3. Let $\vec{a_2}'$ be the error vector $\vec{e} = \vec{a_2} -$ projection on A of $\vec{a_2}$, and unit-ize.
4. Repeat as necessary. Project vector $\vec{a_j}$ onto the matrix A with $\vec{a_1}', \vec{a_2}', \ldots, \vec{a_{j-1}}'$ as its columns.
5. Let $\vec{a_j}'$ be the error vector $\vec{e} = \vec{a_j} -$ projection on A of $\vec{a_j}$, and unit-ize.

Here's a formal algorithm:

1. Let $\vec{a_1}' = \frac{\vec{a_1}}{||\vec{a_1}||}$. Let A have $\vec{a_1}'$ as its only column.
2. Let $\vec{p} = A(A^TA)^{-1}A^T\vec{a_2}$. Let $\vec{e} = \vec{a_2} - \vec{p}$. Let $\vec{a_2}' = \frac{\vec{e}}{||\vec{e}||}$.
3. Repeat as necessary. Let A be the matrix with columns $\vec{a_1}', \ldots, \vec{a_{j-1}}'$.
4. Let $\vec{p} = A(A^TA)^{-1}A^T\vec{a_j}$. Let $\vec{e} = \vec{a_j} - \vec{p}$. Let $\vec{a_j}' = \frac{\vec{e}}{||\vec{e}||}$.

Now we'll find an orthogonal basis for the vector space spanned by vectors $\vec{a_1} = (1, 0, 0)$ and $\vec{a_2} = (1, 0, 1)$ and $\vec{a_3} = (0, 1, 1)$.

We will find an orthogonal basis for the vector space spanned by vectors $\vec{a_1} = (1,0,0)$ and $\vec{a_2} = (1,0,1)$ and $\vec{a_3} = (0,1,1)$.

1. Find $\vec{a_1}'$ and find A.

2. Find the projection $\vec{p}$ of $\vec{a_2}$ on the column space of A, then $\vec{e}$, then $\vec{a_2}'$.

3. Let A be the matrix with columns $\vec{a_1}', \vec{a_2}'$.

4. Find the projection $\vec{p}$ of $\vec{a_3}$ on the column space of A, then $\vec{e}$, then $\vec{a_3}'$.

5. Let Q be the matrix whose columns are $\vec{a_1}', \vec{a_2}', \vec{a_3}'$. This is an *orthogonal matrix*. Check that $Q^T Q = I$.