THREE BOUNDS ON THE INDEPENDENCE NUMBER OF A GRAPH

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ABSTRACT. Three bounds are proved for the independence number α of a nontrivial connected graph: (1) $\alpha \leq n - \frac{C}{2} - \frac{1}{2}$, (2) $\alpha \geq r + \frac{\ell}{2} - 1$, and (3) $\alpha \leq n - \frac{M}{2} - \frac{1}{2}$, where n is the number of vertices of the graph, C is the number of cut vertices, r is the radius, ℓ is the number of pendants, and M is the median degree. The second and third bounds are new. Equality is characterized for the first and third bounds.

1. INTRODUCTION

The *independence number* $\alpha = \alpha(G)$ of a graph G is the cardinality of a largest set of mutually non-adjacent vertices. It is a widely-studied NP-hard graph invariant. Finding bounds for the independence number has been of continuing interest. Lovász has written, "Deriving sharper and sharper upper bounds, more and more insight could be gained into the nature of independence number (a procedure vaguely reminiscent of the expansion of a function into, say, a Fourier series)" [10]. Computational experiments suggest that the Lovász theta function [9] is usually as good or better than any other efficiently computable upper bound. No lower bound is equally good and, in fact, for many graphs (even small graphs) no efficiently computable lower bound predicts the independence number. More than forty efficiently computable bounds for the independence number of a graph can be found in the compilation at `http://independencenumber.wordpress.com`; further non-efficiently computable bounds can be found in [13].

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In what follows, we state and prove three bounds for the independence number of a nontrivial connected graph, describe their origins, and present corollaries. Two of the bounds are new. Equality is characterized for two of the bounds; in the remaining case, an infinite sharp class of graphs is given. Throughout we use $n = n(G)$ for the number of vertices of the graph G. All graphs are considered to be simple and finite.

2. Cut-Vertices Upper Bound

A vertex v is a *cut-vertex* of G if the graph $G - v$, formed by deleting v and all edges incident to v, has more components than G does. The following bound parallels both a cut-vertices lower bound published by the second author [11] as well as a cut-edges bound published by the second author and collaborators [12]. Several proofs of the bound have been communicated to the authors. In particular, Fajtlowicz proved a statement which is equivalent in the case of trees (see [1, G.pc #1]). The proof we give is new and leads to our characterization of the case of equality.

We begin by defining the family of graphs where equality holds in the theorem. A *pendant vertex* (or *pendant*) is a vertex incident to a single edge. A *branch point* is a vertex of degree at least three.

Definition 2.1. *We say that T is an* odd tree *if and only if it is an odd order path or the distance between each pendant vertex and each branch point is odd.*

Theorem 2.2. *Let G be a nontrivial connected graph with independence number α and C cut-vertices. Then*

$$\alpha \leq n - \frac{C}{2} - \frac{1}{2} \qquad (1)$$

and equality holds if and only if G is an odd tree.

Proof. First note that the theorem is true for the connected graph on two vertices, so we may assume that $n \geq 3$. Let T be a spanning tree of G with e' edges and corresponding invariants α' and C'. Let I be a maximum independent set in T containing all of the $\ell = n - C'$ pendant vertices (if a tree has at least three vertices and a maximum independent set does not contain one of the pendants then it must contain its neighbor—these can then be interchanged). Now, let E be the number of edges in T incident to I. Note that since all pendants are in I, and every non-pendant in I has degree at least two, $E \geq 2(\alpha' - \ell) + \ell$. Since $\alpha \leq \alpha'$ and $C \leq C'$, we have:

$$n - 1 = e' \geq E \geq 2(\alpha' - \ell) + \ell = 2\alpha' - \ell = 2\alpha' - n + C' \geq 2\alpha - n + C. \qquad (2)$$

From this we get the desired inequality.

Concerning the case of equality, note that if $\alpha = n - \frac{C}{2} - \frac{1}{2}$, then the first and last expressions from Inequality 2 are equal and so equality holds throughout. In particular, $n - 1 = E = 2(\alpha' - \ell) + \ell$. From this we conclude that for every spanning tree T, all vertices in I have degree one or degree two and all edges of T are incident to a vertex in I. Then I and $T \setminus I$ is a bipartition where all branch points are in $T \setminus I$. Clearly then, the distance from every pendant to every branch point is odd and so T is an odd tree. Moreover, since every cut-vertex of G is a cut-vertex of each of its spanning trees and $C = C'$, the sets of cut-vertices of T and G are identical.

Suppose that G and T were not the same graph. Let xy be an edge of G which is not in T. Then xy is contained in a unique cycle R of the graph $H = T + xy$. Since pendants of T are its only non-cut-vertices, all but at most two vertices of R are cut-vertices of H. Furthermore, the cut-vertices of T which belong to R must be branching points of T (because they are also cut-vertices of H). If there are at least two non-cut vertices of R, they must be consecutive on R. Therefore we have two adjacent branching points of T on R, which is a contradiction because the distance from one of them to a pendant would be even, or R is a triangle with the edge xy joining the two pendants x and y. This is also a contradiction because the independence number of T is more than the independence number of G and these have already been shown to be equal by the collapse of Inequality 2.

Conversely, suppose that G is an odd tree and we will show $\alpha = n - \frac{C}{2} - \frac{1}{2}$. If it is an odd path, then it is easily verified that equality holds. Proceeding by induction on n, assume the implication is true for all odd trees with less than n vertices. Form the graph G' by deleting a pendant vertex and its unique degree two neighbor w, if such a pendant exists. Then, $\alpha' = \alpha - 1$, $C' = C - 2$, and $n' = n - 2$ yielding $\alpha = n - \frac{C}{2} - \frac{1}{2}$ as required (by the inductive hypothesis since G' is still an odd tree). Otherwise, no such pendant exists and then we form the graph G' by deleting a pendant vertex whose unique neighbor is a branch point. Then, $\alpha' = \alpha - 1$, $C' = C$, and $n' = n - 1$ yielding $\alpha = n - \frac{C}{2} - \frac{1}{2}$ as required, using the inductive hypothesis since G' is still an odd tree. □

3. Radius and Pendants Lower Bound

The *eccentricity* of a vertex of a connected graph is the maximum distance from that vertex to any other. The *radius* r of a connected graph is defined to be the minimum eccentricity of all of the vertices. In the 1980's Fajtlowicz's computer program Grafitti conjectured that $\alpha \geq r$ for any connected graph. This result is a consequence of the Induced Path Theorem (namely, that every graph has an induced path with at least $2r - 1$ vertices) which was proved in [5]; the result was also proved independently in [7], and is implied by a more general result in [6].

Graffiti later conjectured that

$$\alpha \geq r + \rho - 1, \tag{3}$$

where ρ denotes the *path covering number*, the minimum number of vertex disjoint paths that can cover all the vertices of the graph. Since Lovász noted (in [9]) that ρ is also a lower bound on independence number (but, like r, the difference between this invariant and α can be arbitrarily large), this conjecture seemed of interest. DeLaVina, Fajtlowicz, and Waller then found a family of counterexamples (described in [3]). After being informed of a counterexample to this conjecture, Graffiti made the following two conjectures.

Conjecture 3.1. *(Graffiti) If a graph G is connected then*

$$\alpha \geq \left\lfloor \frac{r}{2} \right\rfloor + \rho.$$

Conjecture 3.2. *(Graffiti) If a graph G is connected then*

$$\alpha \geq r + \frac{\rho - 1}{2}.$$

Both of these conjectures were proven for trees but remain open in the general case [3].

The main result is that $\alpha \geq r + \frac{\ell}{2} - 1$ for any connected graph with ℓ pendant vertices. Notice that for trees (or any other graph with at least two pendants), this is an improvement on $\alpha \geq r$. It will be shown that Conjecture 3.2, restricted to trees, follows as a corollary to this new bound.

The concept of an r-ciliate and the following theorem of Fajtlowicz provide very useful tools for investigating the relationship of the radius and independence number of a graph.

Definition 3.3. *An* r-ciliate *is a cycle with $2q$ $(q \geq 1)$ vertices and appended to each of these vertices is a path with $r - q$ vertices. They are denoted $C_{2q,r-q}$ (see Figure 1).*

Definition 3.4. *A connected graph G is* radius-critical *if, for any non-cut vertex v, the subgraph $G - v$ formed by deleting v and the edges incident to it has radius less than r.*

The radius of an r-ciliate is r. It is easy to see that r-ciliates are bipartite and that $\alpha(C_{2q,r-q}) = \frac{n(C_{2q,r-q})}{2} = q(r - q + 1)$. In the case where $q = 1$, the cycle is degenerate and identical to the path on two vertices. In this case, the r-ciliate is a path on $2r$ vertices. Clearly, $r \geq q$. The extreme cases are where $q = 1$ and $r = q$. In the latter case the r-ciliate is a cycle on $2r$ vertices.

Theorem 3.5. *(Fajtlowicz, [6]) A connected graph with radius r is radius-critical if and only if it is an r-ciliate.*

This result implies that every connected graph has an induced r-ciliate, and is the foundation for the characterization of those connected graphs

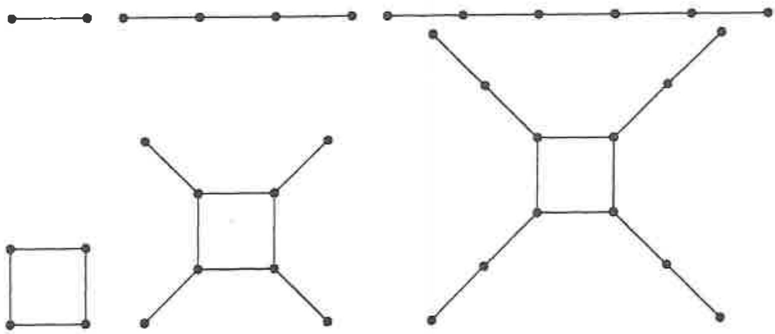

FIGURE 1. In the top row, from left to right, are the r-ciliates $C_{2q,r-q}$ with $q = 1$, and $r = 1, 2$ and 3, respectively. In the second row are the r-ciliates $C_{2q,r-q}$ with $q = 2$, and $r = 2, 3$ and 4, respectively.

whose independence number equals its radius [4]. An induced r-ciliate of a connected graph G can be found by removing non-cut vertices until the remaining (connected) subgraph is radius-critical. The result of this process is not unique and we call the result "an" r-ciliate of G. Theorem 3.5 then guarantees that every connected graph with radius r has an r-ciliate as an induced subgraph.

Theorem 3.6. *If G is a connected graph with radius r, independence number α, and ℓ pendant vertices, then*

$$\alpha \geq r + \frac{\ell}{2} - 1. \tag{4}$$

Proof. Let G be a connected graph, and P be the set of pendant vertices. Let $C_{2q,r-q}$ be an r-ciliate of G. Recall that $r \geq q \geq 1$.

$C_{2q,r-q}$ is bipartite with $2q(r-q+1)$ vertices. Let $\{B, W\}$ be a bipartition of $C_{2q,r-q}$. P is the set of pendant vertices in G. Let P_B and P_W be the set of pendant vertices adjacent to vertices in B and W, respectively. Let P' be the set of pendants which are not included in either of these sets and which do not belong to $C_{2q,r-q}$; so $P' = P \setminus (P_B \cup P_W \cup C_{2q,r-q})$, and $P = P_B \cup P_W \cup P'$.

There are three cases to consider.

Case I, $q = r$. In this case the ciliate $C_{2q,r-q}$ is a cycle and has no pendants. We can assume that $|P_B| \geq |P_W|$. So $|P_B \cup P'| \geq \frac{|P|}{2} = \frac{\ell}{2}$. Note that $W \cup P_B \cup P'$ is an independent set in G. So $\alpha \geq |W \cup P_B \cup P'| \geq |W| + |P_B \cup P'| \geq \frac{n(C_{2q,r-q})}{2} + \frac{|P|}{2} = r + \frac{\ell}{2} \geq r + \frac{\ell}{2} - 1$, which was to be shown.

In the following two cases the ciliate $C_{2q,r-q}$ has pendants. Some of these pendants may be pendants in G and some may not. Let $P'_B = P \cap B$. So $P'_B \subseteq P_W$. Similarly let $P'_W = P \cap W$. So $P'_W \subseteq P_B$. So $P = P_W \cup P_B$. We can assume that $|P_W \cup P'_B \cup P'| \geq |P_B \cup P'_W \cup P'|$. So $|P_W \cup P'_B \cup P'| \geq \frac{|P|}{2} = \frac{\ell}{2}$. Note too that $B \cup P_W \cup P'_B \cup P'$ is an independent set.

Case II, $r > q > 1$. So, in this case, we have

$$\alpha \geq |B \cup P_W \cup P'_B \cup P'| = |B \setminus P'_B| + |P_W \cup P'_B \cup P'| \geq |B| - |P'_B| + \frac{|P|}{2}$$

$$\geq \frac{n(C_{2q,r-q})}{2} - q + \frac{\ell}{2} = q(r-q+1) - q + \frac{\ell}{2} \geq (r-1) + \frac{\ell}{2},$$

which was to be shown. The last inequality follows from the fact that $q(r-q+1) - q \geq r-1$, for real numbers r and q with $r > q > 1$.

Case III, $q = 1$. In this case the r-ciliate is a path, and $n(C_{2q,r-q}) = 2r$. Since there is at most one pendant of G in B, $|P'_B| \leq 1$.

Now it follows,

$$\alpha \geq |B \cup P_W \cup P'_B \cup P'| = |B \setminus P'_B| + |P_W \cup P'_B \cup P'| \geq$$

$$|B| - |P'_B| + \frac{|P|}{2} \geq \frac{n(C_{2q,r-q})}{2} - 1 + \frac{\ell}{2} = r - 1 + \frac{\ell}{2},$$

which was to be shown. □

The bound is sharp for the ciliates $C_{2q,1}$. These graphs have $\alpha = 2q$, $r = q+1$, and $2q$ pendant vertices.

Lemma 3.7. *For any tree with ℓ pendants and path covering number ρ,*

$$\ell \geq \rho + 1.$$

Proof. The truth of the statement can easily be checked for small trees. Assume the statement is true for trees with n vertices and suppose that T is a tree with $n+1$ vertices. If T has a vertex which is adjacent to more than one pendant, then let v be one of these pendants and $T' = T - v$. Clearly, $\rho(T') \geq \rho(T) - 1$ and $\ell(T') = \ell(T) - 1$. Since $\ell(T') \geq \rho(T') + 1$ is assumed to be true, $\ell(T) \geq \rho(T) + 1$ now follows immediately.

If no vertex is adjacent to more than one pendant and v is an endpoint of a longest path of T, then v is necessarily adjacent to a vertex of degree 2. Again we let $T' = T - v$, note that $\rho(T') = \rho(T)$, $\ell(T') = \ell(T)$, and the result follows directly from the inductive hypothesis. □

The following proposition follows immediately from Theorem 3.6 and Lemma 3.7 and gives a new proof for Graffiti's Conjecture 3.2 for trees.

Proposition 3.8. *For any tree with radius r and path covering number ρ,*

$$\alpha \geq r + \frac{\rho - 1}{2}.$$

4. Median Degree Upper Bound

The median degree of a graph is defined in terms of the degree sequence $d_1 \le d_2 \le \ldots \le d_n$ of the graph. The *median degree* M is the middle degree if n is odd (that is, $M = d_{\frac{n+1}{2}}$) and the average of the middle degrees if n is even (that is, $M = \frac{1}{2}(d_{\frac{n}{2}} + d_{\frac{n}{2}+1})$). DeLaVina's Grafitti.pc program made the following conjecture. See [2] for details about the program, and [1] for a list of other conjectures of the program.

Theorem 4.1. *If G is a nontrivial connected graph with independence number α and median degree M then*

$$\alpha \le n - \frac{M}{2} - \frac{1}{2}. \tag{5}$$

Equality holds in this bound if and only if G is a star.

Proof. We will proceed in two cases. Suppose first that $\alpha \le \frac{n}{2}$. In this case, it is sufficient to show that $\frac{n}{2}$ is bounded above by the right hand side of the inequality in the theorem. To this end, we first observe that $M \le n-1$, from which we deduce,

$$\frac{n}{2} \le n - \frac{M}{2} - \frac{1}{2}.$$

Hence the result follows.

Now assume that $\alpha > \frac{n}{2}$. Let I be a maximum independent set. Each vertex in I has degree at most $n - \alpha$. Since more than half of the vertices are in I, it is clear that $M \le n - \alpha$. Thus,

$$\alpha \le n - M \le n - \frac{M}{2} - \frac{1}{2},$$

which settles this case and proves the first part of the claim.

Now we turn to the case of equality in the bound. Clearly if a graph is a star then equality holds. Suppose then that G is a nontrivial connected graph and $\alpha = n - \frac{M}{2} - \frac{1}{2}$. Since G is a nontrivial connected graph the degree of each vertex is at least one and $M \ge 1$. There are two cases to consider.

First assume that $\alpha < \frac{n}{2}$. So $n - \frac{M}{2} - \frac{1}{2} < \frac{n}{2}$. Then $n-1 < M$, which is impossible.

So $\alpha \ge \frac{n}{2}$. Let I be a maximum independent set in G. Then, for every $v \in I$, the degree of v is no more than $n - \alpha$. Since $|I| \ge \frac{n}{2}$, it follows that $M \le n - \alpha = n - (n - \frac{M}{2} - \frac{1}{2}) = \frac{M}{2} + \frac{1}{2}$. It follows that $M = 1$ and, hence, $\alpha = n - 1$. So G is a star. $\square$

It should be mentioned that this median bound can be much better than, for instance, the minimum degree upper bound ($\alpha \le n - \delta$, which is similar in form). An example is a graph K_n and a single vertex v connected to some number of the vertices of K_n: whenever $n > d(v)$ the median bound will be better than the minimum degree bound.

The theorem can be improved for connected graphs which are not stars. Before we state this improvement we define two families of graphs where equality holds.

Definition 4.2. *A* complete split graph *is a graph whose vertices can be partitioned into an independent set I and a clique C such that every vertex in I is adjacent to every vertex in C. $CS(m,n)$ denotes the unique complete split graph whose vertices can be partitioned into an independent set I with $|I| = m$ and a clique C with $|C| = n$.*

Definition 4.3. *A* nova *is a connected graph with $n \geq 4$, $\alpha = n-2$, and $M = 2$.*

This family of graphs includes graphs with two central vertices connected to each of a circle of any number of vertices (hence the name) and also includes the graphs $CS(n-2, 2)$. For our purposes, the important fact about novas is that they are *König-Egerváry (KE) graphs* : graphs where the independence number α and the matching number μ sum to the number of vertices of the graph. Maximum independent sets in KE graphs can be identified efficiently [8]. To see that a nova G is a KE graph, consider the following argument. Let I be a maximum independent set and let x and y be the remaining vertices. Suppose there is no matching of the vertices to a pair of vertices in I. Then x and y can only be adjacent to the same vertex. Since the graph is assumed to be connected, it has three vertices, contradicting the definition of a nova.

Theorem 4.4. *If G is a connected non-star with independence number α and median degree M then*

$$\alpha \leq n - \frac{M}{2} - 1. \tag{6}$$

Equality holds in this bound if and only if the graph is a nova, a $CS(k, k+1)$, or $CS(3,3)$.

Proof. Since G is connected and not a star, we have $\alpha \leq n-2$. If $M = 1$, we are done. So we may assume that $M \geq 2$. There are two cases to consider.

First, assume that $\alpha \leq \frac{n}{2}$. It is easy to check that $M \neq n-1$; thus $M \leq n-2$ and $\frac{n}{2} \leq n - \frac{M}{2} - 1$, proving the result.

Second, assume that $\alpha > \frac{n}{2}$. Let I be a maximum independent set. Each vertex in I has degree at most $n-\alpha$. Since more than half of the vertices are in I, it is clear that $M \leq n-\alpha$. Thus, since $M \geq 2$, $\alpha \leq n-M \leq n-\frac{M}{2}-1$,

Now we turn to the case of equality in the bound. It is easy to check that if a graph is a nova, $CS(k-1,k)$, or $CS(3,3)$, then equality holds.

Suppose then that G is a connected non-star with $\alpha = n - \frac{M}{2} - 1$. There are three cases to consider.

In the first case, assume that $\alpha > \frac{n}{2}$. Let I be a maximum independent set. Then, for every $v \in I$, the degree of v is no more than $n - \alpha$. Since $|I| > \frac{n}{2}$, it follows that $M \leq n - \alpha = n - (n - \frac{M}{2} - 1) = \frac{M}{2} + 1$. This implies that $M \leq 2$ and, hence, that $M = 2$ and $\alpha = n - 2$. If $n \geq 4$ then, by definition G is a nova. If $n = 3$ then G is a complete graph, which is also given as $CS(1, 2)$.

In the second case, assume that $\alpha = \frac{n}{2}$. It follows that n is even, the median degree is the average of the middle degrees, $n - \frac{M}{2} - 1 = \frac{n}{2}$ and, hence, $M = n - 2$. Let I be a maximum independent set. So the median degree M is no more than the average of the largest degree in I and $n - 1$. Since the largest degree in I is no more than $\frac{n}{2}$, we get $M \leq \frac{1}{2}(\frac{n}{2} + n - 1)$ and, hence, $n \leq 6$. Since $M \geq 2$, we only consider graphs where $n = 4$ (and, hence $M = 2$ and $\alpha = 2$) or $n = 6$ (and $M = 4$ and $\alpha = 3$). When $n = 4$ these graphs are novas. In the second case, let I be a maximum independent set. So $|I| = 3$. These vertices can have degree at most 3. Since $M = 4$ it follows that each vertex in $V \setminus I$ must have degree 5. So these vertices form a clique and each of them is adjacent to every vertex in I. The graph is $CS(3, 3)$.

In the third case, assume that $\alpha < \frac{n}{2}$. This implies that $\alpha = n - \frac{M}{2} - 1 < \frac{n}{2}$. So $n - 2 < M$ and, since $M < n$, we have $M = n - 1$. Substituting again we get $2\alpha = n - 1$. Let I be a maximum independent set in G. Note that, since $M = n - 1$ and I is an independent set, it follows that every vertex in $V \setminus I$ has degree $n - 1$. So $G[V \setminus I]$, the graph induced on $V \setminus I$, is a complete subgraph of G and G is the complete split graph $CS(\alpha, \alpha + 1)$. □

It should be emphasized that equality in Theorem 4.4 can be checked efficiently. This is largely a consequence of the previously mentioned facts that the property of being König-Egerváry can be checked efficiently and maximum independent sets in these graphs can be found efficiently [8]. It was argued above that novas are König-Egerváry graphs. Given a graph, first check if it is a König-Egerváry graph. If it is find a maximum independent set I. If $|I| = n - 2$, then find the median degree. If $M = 2$, then the graph is a nova.

If the graph is not a nova, then it may be a $CS(k - 1, k)$. These graphs are "almost" König-Egerváry graphs: removing any vertex from the clique on k vertices yields $CS(k - 1, k - 1)$ which is a König-Egerváry graph. For each vertex v adjacent to each of the other vertices check if the graph formed by removing v is König-Egerváry. If it is, find a maximum independent set I, the complement C, and whether this graph is $CS(k - 1, k - 1)$. If it is then the original graph is $CS(k - 1, k)$. At most $\frac{n+1}{2}$ vertices must be tested in order to make a determination.

5. Open Problems

Here is a summary, for the reader's convenience, of problems mentioned in the text that remain open. Recall that r is the radius, ρ is the path covering number, and ℓ is the number of pendant vertices.

(1) Graffiti's Conjecture 3.1: If a graph G is connected then $\alpha \geq \lfloor \frac{r}{2} \rfloor + \rho$. It is true for trees but remains open in the general case.
(2) Graffiti's Conjecture 3.2: If a graph G is connected then $\alpha \geq r + \frac{\rho - 1}{2}$. It is true for trees but remains open in the general case.
(3) Characterize those graphs where $\alpha = \frac{\ell}{2} + r - 1$. Some examples were given above, but the general problem remains open.

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