Graffiti.pc on the 2-domination number of a graph

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Abstract

The k-domination number $\gamma_k(G)$ of a simple, undirected graph G is the order of a smallest subset D of the vertices of G such that each vertex of G is either in D or adjacent to at least k vertices in D. In 2010, the conjecture-generating computer program, Graffiti.pc, was queried for upperbounds on the 2-domination number. In this paper we prove new upper bounds on the 2-domination number of a graph, some of which generalize to the k-domination number.
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Definitions and introduction

Let G be a finite simple graph on vertex set $V(G)$ and edge set $E(G)$. A subset D of the vertices of a graph G is a *k-dominating set* if every vertex of the graph is either in D or adjacent to at least k vertices of D. The *k-domination number* of a graph G is the order of a smallest k-dominating set, which we denote by $\gamma_k(G)$. A subset of the vertices is an *independent set* if no two vertices in the subset are adjacent. The *independence number* of graph G, denoted $\alpha(G)$, is the order of a largest independent set. The intersection of all largest independent sets is called the *independence-core* of the graph and its order the *core size*, which we denote by $\alpha_c(G)$. A subset

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S of the vertices of G is *k-independent* if the subgraph induced by S has maximum degree less than k. The *k-independence number* α_k is the order of a largest k-independent set. Observe that a 1-independent set is simply an independent set, and that a 1-dominating set is simply a dominating set.

These generalized concepts of independent and dominating sets to k-independent and k-dominating sets, respectively, originated in a 1985 paper [12] of Fink and Jacobson, in which they conjectured that the k-domination number of graph was not more than the k-independence number. In the same year the conjecture was proven by Favaron [13]. Since then many papers have appeared presenting bounds on the k-domination number, sometimes bounds on only the 2-domination number and in lesser numbers one finds some papers presenting bounds on the k-independence number. In this paper, we mention many of those results and address how they relate to our results.

In 2010 Graffiti.pc was queried for upperbounds on the 2-domination number of connected graphs. We refer the reader to [9] and [10] for a description of the heuristic used in this query and for a comparison to Fajtlowicz's Graffiti program. Here we simply note that a product of a query of Graffiti.pc, in general, is a list of conjectured noncomparable relations (in this case upperbounds) for a chosen combination of invariants (in this case simply, the 2-domination number). On the webpage *Written on the Wall II* one finds almost 50 conjectures on the 2-domination number (see [11] starting with number 382), many of which are settled in this paper and some generalized to results on the k-domination number. In our closing section, we present some open conjectures and some partial results.

Independence and k-domination

The following Theorem of Favaron (see [13]) was motivated by the conjecture of Fink and Jacobson announced in [12].

Theorem 1. *(Favaron) For any graph G and any positive integer k, every k-independent set D for which $k|D| -| E(G[D])|$ is a maximum is a k-dominating set of G.*

Corollary 2. *(Favaron) For any graph G and any positive integer k, $\gamma_k(G) \leq \alpha_k(G)$.*

Next note that a recent application of Theorem 1 due to Blida, Chellali, Favaron and Meddah in [2] yields interesting corollaries.

Theorem 3. *(Blida et.al.) For any graph G and any positive integers j and k with $j \leq k$,*

$$\alpha_{k+1}(G) \leq \alpha_j(G) + \alpha_{k-j+1}(G).$$

Corollary 4. *(Blida et.al.) Let G be a graph. Then*

$$\gamma_{k+1}(G) \leq \alpha_k(G) + \alpha(G).$$

Corollary 5. *(Blida et.al.) Let G be a graph. Then*

$$\gamma_k(G) \leq k\alpha(G).$$

Graffiti.pc's 382e in [11] suggested a relation between the 2-domination and independence numbers which now follows as a special case of the corollary to our next theorem.

Theorem 6. *For any graph G and any positive integers a and b,*

$$\gamma_{a+b}(G) \leq \gamma_a(G) + \alpha_b(G).$$

Proof. Let a and b be positive integers. Let D be a minimum a-dominating set of G and let I be a b-independent set of $G[V-I]$ such that I is also a b-dominating set of $G[V-I]$ as in Theorem 1. Then every vertex in neither D nor I is adjacent to at least a vertices of D and also at least b vertices from I. Hence $D \cup I$ is $(a+b)$-dominating and has order at most $\gamma_a + \alpha_b$. □

Corollary 7. *Let G be a graph and let k be a positive integer. Then*

$$\gamma_{k+1}(G) \leq \gamma_k(G) + \alpha(G).$$

Proof. Let $a = k$ and $b = 1$. Then this follows immediately from Theorem 6 and $\gamma_1(G) = \gamma(G) \leq \alpha(G)$. □

Take $a = j$ and $b = k - j + 1$ to see that Corollary 4 also follows from Theorem 6 and Corollary 2. Further note that iterating the relation in Theorem 6 also yields Corollary 5.

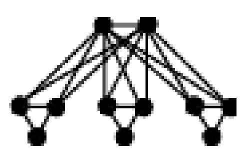

Figure 1: $\gamma_2 = 5, 2\alpha - \alpha_c = 5$ and $\alpha_2 = 6$

One of the Graffiti.pc conjectures that we settled almost immediately was that $\gamma_2(G) \leq 2\alpha(G) - \alpha_c(G)$, which suggests an improvement on the bound of Corollary 5 for $k = 2$ and also provides an immediate sufficient condition for $\gamma_2(G) \leq \alpha(G)$.(Note that various bounds on the core size have been obtained see for instance Hammer et. al. [15], Levit and Mandrescu [20], and Boros et al. [1].) Upon learning of Corollary 2 we naturally wondered if $\alpha_2(G) \leq 2\alpha(G) - \alpha_c(G)$, however, the graph in Figure 1 shows that this is not the case. Still, inspired by Favaron's result on the existence of a k-independent set that is also k-dominating as in Theorem 1, we realized that our original proof of Graffiti.pc's conjecture also suggested a set with both properties (for $k = 2$) and thus this fact is presented separately.

Theorem 8. *Let G be a graph with I a maximum independent set. Then there is a 2-independent set that is 2-dominating and contains I.*

Proof. Let I be a maximum independent set. Let D_2 be the set vertices of $V - I$ that are adjacent to at least two vertices in I. Next, let J be a maximal independent set in the subgraph induced by $V - I - D_2$. To see that $I \cup J$ is a 2-dominating set suppose that $x \in V - (I \cup J)$. If x is in $V - (I \cup J) - D_2$, then x is adjacent to at least one vertex in J and to at least 1 vertex in I. Thus $I \cup J$ is 2-dominating set. Lastly, to see that $I \cup J$ is 2-independent, note that a vertex in J has at most one neighbor in I and no two vertices in J are adjacent to a common vertex without contradicting that I is maximum. $\square$

Theorem 9. *Let G be a graph. Then*

$$\gamma_2(G) \leq 2\alpha(G) - \alpha_c(G).$$

Proof. Let I be a maximum independent set. Let D_2 be a the vertices of $V - I$ that are adjacent to at least two vertices in I and I_c be the independence core. Observe that if a vertex $x \in V - I$ is adjacent to a vertex w in I_c, then x must be adjacent to 2 vertices of I, otherwise $(I - \{w\}) \cup \{x\}$ is maximum independent set and we contradict that w is in the independence core I_c. Thus, $N(I_c) \subseteq D_2$, and when we let J be a maximal independent set in the subgraph induced by $V - I - D_2$, we know by the proof of Theorem 8 that $I \cup J$ is a 2-dominating. Moreover, since I and J are disjoint, it is clear that $|J| + |I_c| \leq \alpha(G)$. Now $\gamma_2 \leq |I \cup J| = |I| + |J| \leq |I| + (|I| -| I_c|) \leq 2\alpha(G) - \alpha_c(G)$. $\square$

The following immediate corollary to Theorem 9 was previously observed by Blidia, Chellali and Volkmann in [4], where they also prove that for a block graph G, $\gamma_2(G) \geq \alpha(G)$, thereby generalizing an earlier result of Blidia, Chellali and Favaron in [3] for trees.

Corollary 10. *Let G be a graph with a unique maximum independent set. Then*

$$\gamma_2(G) \leq \alpha(G).$$

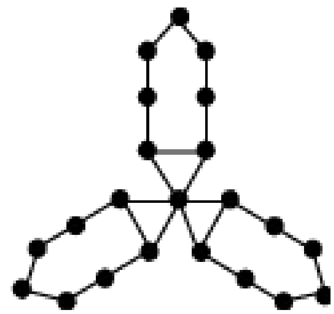

Figure 2: $n(G_4) = 22$, $\alpha(G_4) = 9$ and $\gamma_2(G_4) = 9$

To see that the relation in Theorem 9 is sharp let $k \geq 2$, and begin with three copies of C_{2k-1} and one K_1. In each copy of C_{2k-1} join the endpoints of one edge to the K_1. Call the resulting graph, G_k, and observe that it has $1+3(2k-1) = 6k-2$ vertices, $\alpha(G_k) = 3(k-1)+1$ and $\gamma_2(G_k) = 3(k-1)+1$. This family of graphs also suggests additional graphs for which $\gamma_2 = \alpha$. See G_4 in Figure 2.

Early in our investigation of the 2-domination number of a graph, we proved another relation between the 2-domination number and the independence number of graph. Namely, we proved that $\gamma_2(G) \leq (n+\alpha(G))/2$, which we found interesting. We do not list it in this section but will have occasion to mention it in the third section of this paper (see Corollary 25). However, note that Graffiti.pc conjectured (number 388 in [11]) that in such a bound one need only consider the independence number of the subgraph induced by the vertices of degree at most two, which is stated below and proven in [7].

Theorem 11. *(DeLaVina, Pepper, Vaughan) Let G be a graph on $n \geq 3$ vertices and S_2 the set of vertices of degree at most 2. Then*

$$\gamma_2(G) \leq \frac{n + \alpha(G[S_2])}{2}.$$

Inspired by two of Graffiti.pc's conjectures (numbers 383a and 383b in [11]) that also related the independence number to the 2-domination number, we present a more general statement next.

Theorem 12. *Let G be an n-vertex graph and A_k the set of vertices of degree at least k. Then*

$$\gamma_k(G) \leq n - \alpha(G[A_k]).$$

Proof. Let I_k be a maximum independent set for $G[A_k]$. Then $V - I_k$ is k-dominating, since every vertex in I_k is adjacent to at least k vertices none of which are in I_k. □

A result of Blidia, Chellali and Volkmann in [5] is a corollary to Theorem 12.

Corollary 13. *(Blidia, Chellali and Volkmann) Let G be an n-vertex bipartite graph and let $d(v)$ denote the degree of a vertex v of G. Then*

$$\gamma_k(G) \leq \frac{n + |\{x \in V : d(x) \leq k-1\}|}{2}.$$

Proof. Let A_k be the set of vertices of degree at least k. Then $\{x \in V : d(x) \leq k-1|\} = V - A_k$. Since $\alpha(G[A_k]) \geq \frac{|A_k|}{2}$ for bipartite graphs, Theorem 12 yields

$$\gamma_k(T) \leq n - \alpha(G[A_k]) \leq n - \frac{|A_k|}{2} = \frac{n + |V - A_k|}{2}.$$

□

A special case (when $k = 2$) of our next corollary that follows immediately from Theorem 12 was observed by Blidia, Chellali and Volkmann in [4].

Corollary 14. *Let G be an n-vertex graph with minimum degree $\delta(G)$. If $\delta(G) \geq k$, then*

$$\gamma_k(G) \leq n - \alpha(G).$$

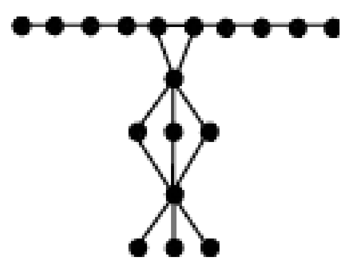

Figure 3: $n = 18$, $\alpha(G[A_2]) = 7$ and $\gamma_2 = 11$

To see that for $k = 2$ the bound in Theorem 12 is sharp for infinitely many graphs, let $m \geq 2$, begin with a $K_{2,m}$ and label the two vertices of degree m as a and b. Then join a to 3 discrete vertices and join b to the two centers of a path on 10 vertices. This graph has $n = m + 15$, $\alpha(G[A_2]) = m + 4$ and $\gamma_2 = 11$ (see Figure 3 for an example.)

Our last corollary to Theorem 12 has been observed in several papers; we note it again since we will have occasion to use it.

Corollary 15. *Let T be an n-vertex tree with L leaves. Then*

$$\gamma_2(T) \leq \frac{n+L}{2}.$$

Proof. The number of vertices of degree at least 2 in T is $n - L$, and $\alpha(G[A_2])$, the independence number of the subgraph induced by the non-leaves, is at least $\frac{n-L}{2}$. Thus, $\gamma_2(T) \leq n - \alpha(G[A_2]) \leq n - \frac{n-L}{2} = \frac{n+L}{2}$. $\square$

A *cut vertex* of a graph is a vertex whose removal from the graph increases the number of components of the graph. The *number of cut vertices* of G is denoted by $\kappa(G)$. Graffiti.pc's number 384a involving the number of cut vertices follows from a straightforward spanning tree argument and the previous corollary.

Proposition 16. *Let G be a connected n-vertex graph. Then*

$$\gamma_2(G) \leq n - \frac{\kappa(G)}{2}.$$

Proof. Let T be a spanning tree of G with L leaves. Since $\gamma_2(G) \leq \gamma_2(T)$ and $\kappa(G) \leq \kappa(T)$, $\gamma_2(G) \leq \gamma_2(T) \leq (n + L)/2 = n - \kappa(T)/2 \leq n - \kappa(G)/2$. $\square$

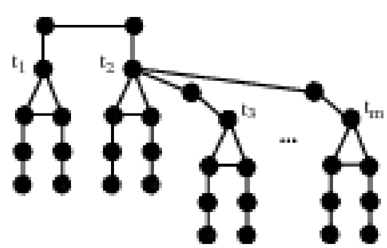

Figure 4: $n = 8m$, $\kappa = 6m$ and $\gamma_2 = 5m$

The graphs in Figure 4 demonstrate that the relation in Proposition 16 is sharp for infinitely many graphs.

Path Covering and 2-domination

A collection of vertex disjoint paths of a graph that partition the vertices of G is a *path covering* of G. The *path covering number* of a graph G is the cardinality of a minimum path covering of the graph and is denoted by $\rho(G)$. Note that $\rho(G) = 1$ if and only if G has a Hamiltonian path.

Our next theorem originated as Graffiti.pc's number 390 in [11]. Note that in 2007 the program also conjectured that the total domination number (the order of a smallest subset of the vertices such that every vertex of the graph is adjacent to a vertex in the subset) is bounded above by the sum of path covering number and the matching number (see [8]). Although the total domination number and 2-domination numbers, in general, are not comparable, Haynes et.al in [16] show that for nontrivial trees $\gamma_t \leq \gamma_2$ and in [6] Chellali extended this to $\gamma_t - c(G) \leq \gamma_2$ for a cactus graph G with $c(G)$ even cycles.

Theorem 17. *Let G be a connected graph. Then*

$$\gamma_2(G) \leq \rho(G) + \mu(G).$$

Proof. Let $\rho = \rho(G)$ and let $P = \{P_1, P_2, ..., P_\rho\}$ be a minimum path covering of G with P_i having n_i vertices. Starting from one end, let M_i be the matching consisting of the edges in odd position along P_i, so that $|M_i| = \lfloor \frac{n_i}{2} \rfloor$. For each i such that $1 \leq i \leq \rho$, we construct a 2-dominating set D_i for P_i such that $|D_i| \leq| M_i| + 1$ as follows. For i such that $n_i = 0$, simply let D_i be the single vertex of P_i and observe that since $|M_i| = 0$, $|D_i| = |M_i| + 1$. For i such that $n_i > 1$, to form D_i we take every other vertex along P_i starting from one end and the vertex at the other end of P_i. If n_i is odd, this works very simply, since $|D_i| = \frac{n_i+1}{2} = \frac{n_i-1}{2} + 1 = |M_i| + 1$. If n_i is even, then $|D_i| = \frac{n_i}{2} + 1 = |M_i| + 1$. This completes the proof, since $\bigcup_{i=1}^{\rho} D_i$ is a 2-dominating set of order at most $\rho + |\bigcup_{i=1}^{\rho} M_i|$ which at most $\rho + \mu(G)$. $\square$

To see that the bound in Theorem 17 is sharp for infinitely many graphs, start with a cycle on $m \geq 3$ vertices. Then join each vertex of the cycle to a pair of discrete vertices, so that the resulting graph has $3m$ vertices; it is not difficult to see that $\rho = m$, $\mu = m$ and $\gamma_2 = 2m$.

Corollary 18. *Let G be a bipartite graph. Then*

$$\gamma_2(G) \leq \rho(G) + \alpha(G).$$

Proof. This follows immediately from Theorem 17, since $\mu(G) \leq \alpha(G)$ for bipartite graphs. $\square$

Lemma 19. *Let G be an n-vertex bipartite graph, let I be a maximum independent set of vertices in G,and $I^c = V(G) - I$. Then there exists a matching, each of whose edges has one endpoint in I and one endpoint in I^c, such that each vertex of I^c is incident to an edge in the matching.*

Proof. Since for bipartite graphs, $\mu(G) = n - \alpha(G)$, and there are $n - \alpha(G)$ vertices in I^c, each vertex of I^c must be saturated by every maximum matching since there are no edges joining vertices of I. Moreover, each edge of a maximum matching must be incident to exactly one vertex in I. □

Theorem 20. *Let G be a bipartite graph on $n \geq 3$ vertices. Then*

$$\gamma_2(G) \leq 2\alpha(G) - \rho(G).$$

Proof. Let I be a maximum independent set of vertices in G containing the leaves of G, and let $I^c = V(G) - I$. By the lemma, there exists a matching, each of whose edges has one endpoint in I and one endpoint in I^c, such that each vertex of I^c is incident to an edge in the matching. Color the edges of this matching red. Likewise, consider a maximum matching M in the subgraph induced by I^c, and color the edges of this matching green. Now form a path covering R of G as follows: if a vertex v of I is not incident to a red edge, then add v to R as a singleton path. If two red edges are adjacent to the same green edge, then add these edges to R as a four-path (note that these edge cannot form a triangle). Finally, add the remaining red edges to R as two-paths. It is easy to see every vertex of G is contained in exactly one path in R.

Next we will color some of the vertices of G as follows: For each green edge, if an endpoint v of the edge is adjacent to a vertex in I^c not incident to a green edge, then color v white, and color its neighbor in I incident to the red edge containing v yellow. If neither endpoint of the green edge is adjacent to a vertex in I^c not incident to a green edge, then arbitrarily choose one endpoint v and color the vertex white, and again color its neighbor in I incident to the red edge containing v yellow. The remaining vertices of I will be colored blue. We will label the set of white vertices W, the set of yellow vertices Y, and the set of blue vertices B. Note that $|W| = |Y|$.

Claim 1. Each four-path in R contains at most one white vertex and at least one blue vertex.

Proof of Claim 1. By way of contradiction, suppose there exists a four-path P in R that contains two white vertices, say a and b, which thus must be joined by a green edge. Therefore a must be adjacent to a vertex $a' \neq b$ of I^c and b must be adjacent to a vertex $b' \neq a$ of I^c neither of which is incident to a green edge. Since G is bipartite, $a' \neq b'$. Moreover, if a' and

b' are adjacent, then M is not maximum, a contradiction. But then we can find a larger matching in the subgraph induced by I^cthan M, by deleting the edge joining a and b from M and in turn adding the edges joining a and a' and b and b', again a contradiction. Now it is easy to see that P must contain a blue vertex as well.

Claim 2. $X = I \cup W$ is a 2-dominating set of G.

Proof of Claim 2. Since X contains I, then the only vertices of G that may not be 2-dominated by X are those vertices of I^c that are adjacent to exactly one vertex in I. Suppose v is such a vertex. If v is incident to a green edge, then either $v \in W$ or v is adjacent to a white vertex. In either case, v is 2-dominated. On the other hand, suppose v is not incident to a green edge. Since v is not a leaf, then v must be adjacent to another vertex a of I^c. If a is incident to a green edge, then a must be colored white and hence v is 2- dominated. If a is not incident to a green edge, then M is not maximum, a contradiction.

Finally, observe by Claim 1 that each path in R must contain at least one blue vertex, hence $|B| \geq |R| \geq \rho(G)$. Therefore, by Claim 2,

$$\begin{aligned}\gamma_2(G) &\leq |X| = |I \cup W| = \alpha(G) + |W| = \alpha(G) + |Y| = \alpha(G) + |I - B| \\ &= \alpha(G) + \alpha(G) - |B| \leq 2\alpha(G) - \rho(G)\end{aligned}$$

□

Neighborhoods and k-domination

Let X be a subset of the vertices of G. The *neighborhood* of X, denoted $N(X)$, is the set of all vertices adjacent to some vertex in X. The *subgraph induced by* X is denoted $G[X]$.

Inspired by several of Graffiti.pc's conjectures we prove that any subset of the vertices is the starting point for a 2-dominating set from which several of Graffiti.pc's conjectures and other known bounds follow as corollaries.

Theorem 21. *Let G be an n-vertex graph, $S \subseteq V$ and $G[N(S) - S]$ the subgraph of G induced by $N(S) - S$. Then*

$$\gamma_2(G) \leq n - |N(S) - S| + \gamma(G[N(S) - S]).$$

Proof. Let D be a smallest dominating set for $G[N(S) - S]$. Observe that D and S are disjoint, and that $S \subseteq V - (N(S) - S)$. To see that $(V - (N(S) - S)) \cup D$ is a 2-dominating set for G, let v be a vertex in $N(S) - S$. Clearly we can assume that v is not also in D. Now, since v is adjacent to at least one vertex in D and to at least one vertex in S, it follows that $(V - (N(S) - S)) \cup D$ is a 2-dominating set. □

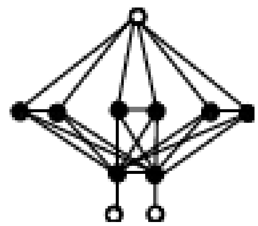

Figure 5: $n = 11$, $\gamma_2 = 4$ and for S the set of white vertices, $\Delta(G[N(S) - S)] = 7$.

The next corollary was inspired by three of Graffiti.pc's conjectures (385a, b and c in [11]). The graph in Figure 5 demonstrates that the bound in Corollary 22 is sometimes sharp.

Corollary 22. *Let G be an n-vertex graph and $S \subset V$ and $G[N(S) - S]$ the subgraph of G induced by $N(S) - S$. Then*

$$\gamma_2(G) \leq n - \Delta(G[N(S) - S]).$$

Proof. By Theorem 21 and the well-known Berge inequality $\gamma(G[N(S) - S]) \leq |N(S) - S| - \Delta(G[N(S) - S]$ one immediately sees that

$$\begin{aligned} \gamma_2(G) &\leq n - |N(S) - S| + \gamma(G[N(S) - S]) \\ &\leq n - |N(S) - S| + |N(S) - S| - \Delta(G[N(S) - S] \\ &\leq n - \Delta(G[N(S) - S]) \end{aligned}$$

□

Our next theorem has interesting corollaries.

Theorem 23. *Let G be an n-vertex graph, P the set of pendant vertices, $S \subseteq V$ such that $P \subseteq S$ and G^* the subgraph induced by the non-trivial components of $G[N(S) - S]$. Then*

$$\gamma_2(G) \leq n - |N(S) - S| + \gamma(G^*).$$

Proof. Let D be a smallest dominating set for G^*. Observe that D and S are disjoint. To see that $(V - (N(S) - S)) \cup D$ is a 2-dominating set for G, let v be a vertex in $N(S) - S$. Clearly we can assume that v is not also in D. If v is not an isolate of $G[N(S) - S]$, then v is adjacent to at least one vertex in D and to at least one vertex in S. On the other hand, observe that by construction, v is not an isolated vertex of G. Thus, if v is isolated in $G[N(S) - S]$, then since the pendant vertices are in S, v must have at least two neighbors in $V - (N(S) - S)$. □

Corollary 24. *Let G be an n-vertex graph and D a dominating set that contains all pendants. Then*

$$\gamma_2(G) \leq \frac{n + |D|}{2}.$$

Proof. Since D is dominating, $N(D) - D = V - D$. It is well known that the domination number of a graph with no isolated vertices is at most half the number of vertices, and so a minimum dominating set of G^*, the subgraph induced by the non-isolates of $G[V - D]$, has order at most $(n - |D|)/2$. Thus, by Theorem 23 we see that $\gamma_2(G) \leq n - |N(D) - D| + \gamma(G^*) = n - (n - |D|) + \gamma(G^*) \leq |D| + (n - |D|)/2 = (n + |D|)/2$. □

Corollary 25. *Let G be a graph on $n \geq 3$ vertices. Then*

$$\gamma_2(G) \leq \frac{n + \alpha(G)}{2}.$$

Proof. Let D be a maximum independent set that contains all pendants. Then $\gamma_2(G) \leq \frac{n+\alpha(G)}{2}$ follows immediately from Corollary 24. □

Next we note that the simple upperbound on γ_2 for bipartite graphs of Fujisawa et. al in [14] (generalizing a result found in [3]) also follows easily as our next corollary. Note that in [14] they characterized the case of equality.

Corollary 26. *(Fujisawa et.al.) Let G be a bipartite graph on $n \geq 3$ vertices. Then*

$$\gamma_2(G) \leq \frac{3}{2}\alpha(G).$$

Proof. This bound follows immediately from Corollary 25, since for bipartite graphs $\frac{n}{2} \leq \alpha(G)$. □

Corollary 27. *Let G be an n-vertex graph and P the set of pendants. Then*

$$\gamma_2(G) \leq \frac{n + \gamma(G) + |P|}{2}.$$

Proof. Let D be a minimum dominating set union the set of all pendants. Then $\gamma_2(G) \leq \frac{n+\gamma(G)+|P|}{2}$ follows immediately from Corollary 24. □

Corollary 28. *(Blidia, Chellali, Volkmann) Let G be an n-vertex graph with minimum degree $\delta(G)$ at least 2. Then*

$$\gamma_2(G) \leq \frac{n + \gamma(G)}{2}.$$

Proof. Since $\delta(G) \geq 2$, the set of pendants P is empty, and $\gamma_2(G) \leq \frac{n+\gamma(G)}{2}$ follows immediately from Corollary 27. $\square$

In general, the domination number and the matching number of a graph are noncomparable, however it is well known that for graphs with no isolated vertices (see [17]) the domination number is at most the matching number. Theorem 23 and the previous observation settle Graffiti.pc's number 397b.

Corollary 29. *Let G be a connected n-vertex graph such that $n \geq 3$ and P the set of pendant vertices. Then*

$$\gamma_2(G) \leq n - |N(P)| + \mu(G[N(P)]).$$

Proof. Since $n \geq 3$ and G is connected, no two pendants are adjacent and so $N(P) - P = N(P)$. Thus for G^* the subgraph induced by the non-isolated vertices of $G[N(P) - P]$, $\mu(G^*) = \mu(G[N(P)])$. Now Theorem 23 (with P as our set) and $\gamma(G^*) \leq \mu(G^*)$ yield $\gamma_2(G) \leq n - |N(P) - P| + \gamma(G^*) = n - |N(P)| + \gamma(G^*) \leq n - |N(P)| + \mu(G^*) = n - |N(P)| + \mu(G[N(P)])$. $\square$

Let S be a subset of the vertices of a graph G. A vertex in $V - S$ with exactly one neighbor in S is called a *private external neighbor* of S. The number of all external private neighbors of S is denote by $pn(S)$.

Proposition 30. *Let G be a graph and $S \subseteq V$. Then*

$$\gamma_2(G) \leq n - |N(S) - S| + pn(V - (N(S) - S)).$$

Proof. Let P' be the set of private external neighbors of $V - (N(S) - S)$. Since $S \subseteq V - (N(S) - S)$ and every vertex that is not in $V - (N(S) - S)$ has at least one neighbor in S, $P' \cup [V - (N(S) - S)]$ is a 2-dominating set and the result follows. $\square$

Our next immediate corollary settles Graffiti.pc's conjecture 397a in [11].

Corollary 31. *Let G be an n-vertex connected graph such that $n \geq 3$ and P the set of pendant vertices. Then*

$$\gamma_2(G) \leq |V - N(P)| + pn(V - N(P)).$$

Proof. Let S be P and observe that $N(P) - P$ is simply $N(P)$. Then this follows from Proposition 30.

$\square$

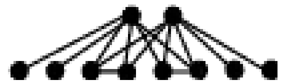

Figure 6: $n = 10$, $|N_2| = 4$ and $\gamma_2 = 6$.

Our final result of this section settles Graffiti.pc's 386a which states that the 2-domination number of a graph is not more than the order of the (set) complement of a largest intersection of neighborhoods of any two vertices. We make the obvious generalization of this bound to k-domination number in the next proposition and note that the graph in Figure 6 demonstrates that the relation is sometimes sharp when $k = 2$.

Proposition 32. *Let $G = (V, E)$ be an n-vertex graph and N_k the largest intersection of neighborhoods of any k vertices of G. Then*

$$\gamma_k(G) \leq n - |N_k|.$$

Proof. If $|N_k| = 0$, then the result follows trivially. So suppose that $|N_k| \geq 1$. Then $V - N_k$ is a k-dominating set, since every $u \in N_k$ is adjacent to k vertices in $V - N_k$. Thus $\gamma_k \leq n - |N_k|$. □

Berge's well known upper bound for the domination number of a graph is a special case of Proposition 32.

Corollary 33. *Let $G = (V, E)$ be an n-vertex graph. Then $\gamma(G) \leq n - \Delta(G)$.*

Conjectures and Partial Results

Recall that the statistical median of a sequence is the middle number when the sequence is ordered and the number of values is odd, and it is the average of the two middle numbers when the number of values is even. Let the degree sequence be ordered in nondecreasing order $d_1 \leq d_2 \leq ... \leq d_{n-1} \leq d_n$. The *upper median* of the degree sequence, denoted by $m(G)$, is defined as the (statistical) median in case the number of vertices is odd and the set $m(G)$ equal to $d_{n/2+1}$ in case the number of vertices is even. Since stars demonstrate than the Berge-like relation involving γ_2, n and Δ does not hold, the following conjecture of Graffiti.pc that $\gamma_2 \leq n - (m(G) - 1)$ seemed of interest.

Conjecture 1. *Let G be a connected n-vertex graph. Then*

$$\gamma_2(G) > n - m(G) + 1,$$

where $m(G)$ is the upper median of the degree sequence of G.

The conjecture remains open in general, but next we prove the slightly stronger relation whenever $\alpha_2(G) > n/2$, which is used to partially settle Conjecture 1 (in particular for bipartite graphs).

Theorem 34. *Let G be an n-vertex graph such that $\alpha_2(G) > n/2$. Then $\alpha_2(G) \leq n - m(G) + 1$, where $m(G)$ is the upper median of the degree sequence of G.*

Proof. Let I be a largest 2-independent set with more than $n/2$ vertices. Then, since each vertex of I has degree at most $n - |I| + 1$ and more than half the vertices are in I, the upper median degree is at most $n - |I| + 1$. Hence, $|I| \leq n - m(G) + 1$. □

Corollary 35. *If G is bipartite, then $\alpha_2 \leq n - m(G) + 1$, where $m(G)$ is the upper median of the degree sequence of G.*

Proof. If $\alpha_2(G) > n/2$, the result follows directly from Theorem 34. So, assume that $\alpha_2(G) = n/2$. In this case, $\alpha_2(G) = \alpha(G) = n/2$. Consequently, both parts of the bipartite graph are maximum independent sets. Whence the maximum degree is at most $n/2$, from which it follows that $n/2$ is at most n minus the maximum degree. Therefore, $\alpha_2(G)$ is at most n minus the maximum degree, from which the result follows. □

Conjecture 2. *Let G be a connected graph. Then*

$$\gamma_2(G) \leq A(G) + 1,$$

where $A(G)$ is the largest integer k such that there exist k vertices whose degree sum is not more than the number of edges of G.

Note that the graph invariant $A(G)$ present in Conjecture 2 was introduced in [21], and called the *annihilation number* of a graph. In [21] and [22] Pepper proved that the annihilation number is an upperbound on the independence number of a graph (which we use in the proof of our next proposition), and in [19] the case of equality for the upperbound was characterized by Larson. Note that it is easily seen that $A(G) \geq (n-1)/2$ and thus when $\gamma_2 \leq n/2 + 1$ the relation in Conjecture 2 follows giving us a partial result for the conjecture. Next we observe two other partial results for Conjecture 2.

Proposition 36. *If graph G has a unique maximum independent set, then*

$$\gamma_2(G) \leq A(G),$$

where $A(G)$ is the largest integer k such that there exist k vertices whose degree sum is not more than the number of edges of G.

Proof. By Corollary 10 and the above mentioned upperbound on $\alpha(G)$ found in [21], we see that $\gamma_2(G) \leq \alpha(G) \leq A(G)$. $\square$

Proposition 37. *Let G be a graph on $n \geq 3$ vertices. Then*

$$\gamma_2(G) \leq \frac{3}{2}A(G) + \frac{1}{2},$$

where $A(G)$ is the largest integer k such that there exist k vertices whose degree sum is not more than the number of edges of G.

Proof. By Corollary 25, $(n-1)/2 \leq A(G)$ and the above mentioned upperbound found in [21], we see that $\gamma_2(G) \leq \frac{n+\alpha(G)}{2} = \frac{n-1}{2} + \frac{\alpha(G)}{2} + \frac{1}{2} \leq A(G) + \frac{A(G)}{2} + \frac{1}{2} = \frac{3}{2}A(G) + \frac{1}{2}$.

$\square$

The *total domination number* of a graph G, denoted $\gamma_t(G)$, is the order of a smallest subset of the vertices such that every vertex of the graph is adjacent to a vertex in the subset. Again we note that in 2007 the program conjectured the similar upperbound for the total domination number involving the annihilation number (see number 298 in [11]), which as far as we know remains open.

Conjecture 3. *Let G be a connected graph. Then*

$$\gamma_t(G) \leq A(G) + 1,$$

where $A(G)$ is the largest integer k such that there exist k vertices whose degree sum is not more than the number of edges of G.

Conjecture 4. *Let G be an n-vertex connected graph. Then*

$$\gamma_2(G) \leq WP(\overline{G}) + 1,$$

where $WP(\overline{G})$ is the largest integer k such that $d_k(G) + k \leq n$.

Note that $WP(\overline{G})$ is called the *Welsh-Powell invariant* (see [24]) of the complement of G.

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